Qualitative Analysis of Delay Partial Difference Equations

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Contents

Preface

This monograph is devoted to a rapidly developing area of the research of the qualitative theory of difference equations. In particular, we are interested in the qualitative theory of delay partial difference equations. The qualitative theory of delay difference equations has attracted many researchers since 1988. The proliferation of this area has been witnessed by several hundreds of research papers and a number of research monographs. It is known that most practical problems are of multiple variables. Therefore, the research of partial difference equations is significant.

Recently, a monograph of partial difference equations has been published by S. S. Cheng. The mathematical modeling of many real-world problems leads to differential equations that depend on the past history in addition to the current state. An excellent monograph of partial functional differential equations has been published by J. Wu in 1996. By the same reason, many mathematicians have been working on the delay partial difference equations. Much fundamental framework has been done on the qualitative theory of delay partial difference equations in the past ten years. And to the best of our knowledge, there has not been a book in the literature presenting the systematical theory on delay partial difference equations so far.

This book provides a broad scenario of the qualitative theory of delay partial difference equations. The book is divided into five chapters. Chapter 1 introduces delay partial difference equations and related initial value problems, and offers several examples for motivation. In Chapter 2, we first discuss the oscillation of the linear delay partial difference equations with constant parameters, where the characteristic equations play an important role, then we present some techniques for the investigation of the oscillation of the linear delay partial difference equations with variable coefficients. Chapter 3 is devoted to the study of the oscillation of the nonlinear delay partial difference equations. In Chapter 4, we consider the stability of the delay partial difference equations. In the last chapter, we introduce some recent works on spatial chaos.

Most of the material in this book is based on the research work carried out by authors and some other experts and graduate students during the past ten years.

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1 Preliminaries

1.1. Introduction

Mathematical computations are frequently based on equations that allow us to compute the value of a function recursively from a given set of values. Such an equation is called a "difference equation." Partial difference equations are types of difference equations that involve functions of two or more independent variables. Such equations occur frequently in the approximation of solutions of partial differential equations by finite difference methods, random walk problems, the study of molecular orbits, dynamical systems, economics, biology, population dynamics, and other fields.

The theory of delay partial differential equations has been studied rigorously recently. Delay partial difference equations can be considered as discrete analogs of delay partial differential equations.

Example 1.1. In order to describe the survival of red blood cells in animals, Wazewska-Czyzeska and Lasota proposed the equation

$$p'(t) = -\delta p(t) + qe^{-ap(t-\tau)}, \tag{1.1}$$

where $p(t)$ is the number of the red blood cells at time t, δ is the rate of death of the red blood cells, q and a are parameters related to the generation of red blood cells per unit time, and τ is the time needed to produce blood cells. If we add one spatial variable to (1.1) and assume that spatial migration is possible, then (1.1) becomes the delay reaction diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = d\triangle p(x,t) - \delta p(x,t) + qe^{-ap(x-\sigma,t-\tau)}, \quad (x,t) \in \Omega \times (0,\infty) \equiv G, \tag{1.2}$$

where d is a positive constant, Ω is a bounded domain in R, where R denotes the set of all real numbers, $\triangle p(x,t) = \partial^2 p(x,t)/\partial x^2$, τ and σ are positive constants.

By means of standard difference method, we replace the second-order partial derivative $\Delta p(x,t)$ by central difference and $\partial p(x,t)/\partial t$ by the forward difference, then under the assumption $p(x_m, t_n) \approx p_{m,n}$, (1.2) becomes the nonlinear delay partial difference equation

$$p_{m+1,n} + p_{m,n+1} - p_{m,n} = -\delta p_{m,n} + qe^{-ap_{m-\sigma,n-\tau}}, \quad (m,n) \in N_0^2, \tag{1.3}$$

where $q, a \in (0,\infty)$, $\delta \in (0,1)$, σ and $\tau \in N_1$, $N_t = \{t, t+1, \dots\}$, $p_{m,n}$ represents the number of the red blood cells at site m and time n.

Example 1.2. Consider the temperature distribution of a "very long" thin rod. We put a uniform grid on the rod and label the grid vertices with integers. Let $u_{m,n}$ be the temperature at the integral time n and the integral position m of the rod. At time n, if the temperature $u_{m-1,n}$ is higher than $u_{m,n}$, heat will flow from the point $m-1$ to m. The change of temperature at position m is $u_{m,n+1} - u_{m,n}$, and it is reasonable to postulate that this change is proportional to the difference $u_{m-1,n} - u_{m,n}$, say, $r(u_{m-1,n} - u_{m,n})$, where r is a positive diffusion rate constant. Similarly, heat will flow from the point $m+1$ to m. Thus, the total effect is

$$u_{m,n+1} - u_{m,n} = r(u_{m-1,n} - u_{m,n}) + r(u_{m+1,n} - u_{m,n}), \quad m \in Z,\ n \in N_0, \tag{1.4}$$

where $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

Such a postulate can be regarded as a discrete form of the Newton law of cooling. If we assume that the rod is semi-infinite, then (1.4) is defined on $(m,n) \in N_0^2$.

In the model (1.4), we assume that heat flow is instantaneous. However, in reality, it takes time for heat to flow from one point m to its neighboring points $m-1$ and $m+1$. Thus a corresponding model is the following delay heat equation:

$$u_{m,n+1} - u_{m,n} = r(u_{m-1,n-\sigma} - u_{m,n-\sigma}) + r(u_{m+1,n-\sigma} - u_{m,n-\sigma}), \tag{1.5}$$

that is,

$$u_{m,n+1} - u_{m,n} = ru_{m-1,n-\sigma} - 2ru_{m,n-\sigma} + ru_{m+1,n-\sigma}. \tag{1.6}$$

In this monograph, we develop the qualitative theory of delay partial difference equations, especially the oscillation theory and the stability theory for delay partial difference equations with two variables. We will introduce some recent results about spatial chaos in the final chapter.

1.2. Initial value problems and initial boundary value problems

In Chapter 2, we will consider the delay partial difference equations of the form

$$A_{m+1,n} + A_{m,n+1} - pA_{m,n} + \sum_{i=1}^{u} q_i A_{m-k_i,n-l_i} = 0, \tag{1.7}$$

where p and q_i are real numbers, k_i and $l_i \in N_0$, $i = 1, 2, \ldots, u$, and u is a positive integer.

Set $k = \max k_i$, $l = \max l_i$, $i = 1, 2, \ldots, u$. The set $\Omega = N_{-k} \times N_{-l} \setminus N_0 \times N_1$ is called the initial domain. A function $\varphi_{i,j}$ defined on Ω is called the initial function. Equation (1.7) together with an initial condition

$$A_{i,j} = \varphi_{i,j}, \quad (i, j) \in \Omega, \tag{1.8}$$

is called an initial value problem.

Using inductive arguments it is easy to see that the initial value problem (1.7) and (1.8) has a unique solution $\{A_{m,n}\}$, $(m, n) \in N_0 \times N_1$. In fact, we rewrite (1.7) in the form

$$A_{m,n+1} = pA_{m,n} - A_{m+1,n} - \sum_{i=1}^{u} q_i A_{m-k_i,n-l_i}, \tag{1.9}$$

and use it to successively calculate $A_{0,1}$, $A_{1,1}$, $A_{0,2}$, $A_{2,1}$, $A_{1,2}$, $A_{0,3}$, $\ldots$. The double sequence $\{A_{m,n}\}$ is unique, and is called a solution of the initial value problem (1.7) and (1.8).

For some partial difference equations, we have to consider the initial condition together with certain boundary value conditions, which is usually the case in partial differential equations.

Example 1.3. Consider the delay parabolic equation

$$\frac{\partial u(x,t)}{\partial t} = a(t)\frac{\partial^2 u(x,t)}{\partial x^2} - q(t)u(x, t - \sigma), \tag{1.10}$$

where $\sigma > 0$ is the delay. Such equations can be used to model problems of population dynamics with spatial migrations. However, in population dynamics where the population density fluctuation in a seasonal manner and settlements are allowed only in concentrated locations, it is more appropriate to consider partial difference equations with delay of the form

$$\Delta_2 u_{i,j} = a_j \Delta_1^2 u_{i-1,j} - q_j u_{i,j-\sigma}, \quad 1 \le i \le n, \ j \ge 0, \tag{1.11}$$

where σ is a nonnegative integer, $a_j, q_j : N_0 \to R$, and

$$\begin{aligned} \Delta_1 u_{i,j} &= u_{i+1,j} - u_{i,j}, \\ \Delta_2 u_{i,j} &= u_{i,j+1} - u_{i,j}, \\ \Delta_1^2 u_{i-1,j} &= \Delta_1(\Delta_1 u_{i-1,j}) = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}. \end{aligned} \tag{1.12}$$

We will assume that $u_{i,j}$ is subject to the conditions

$$\begin{aligned} u_{0,j} + \alpha_j u_{1,j} &= 0, \quad j \geq 0, \\ u_{n+1,j} + \beta_j u_{n,j} &= 0, \quad j \geq 0, \\ u_{i,j} = \rho_{i,j}, \quad -\sigma \leq j &\leq 0,\ 0 \leq i \leq n+1, \end{aligned} \tag{1.13}$$

where $\alpha_j + 1 \geq 0$ and $\beta_j + 1 \geq 0$ for $j \geq 0$.

Given an arbitrary function $\rho_{i,j}$ which is defined on $-\sigma \leq j \leq 0$ and $0 \leq i \leq n+1$, we can show that a solution to (1.11)–(1.13) exists and is unique. Indeed, from (1.11), we have

$$\begin{aligned} u_{i,1} &= a_0\rho_{i+1,0} + (1-2a_0)\rho_{i,0} + a_0\rho_{i-1,0} - q_0\rho_{i,-\sigma}, \quad 1 \leq i \leq n, \\ u_{0,1} &= -\alpha_1 u_{1,1}, \qquad u_{n+1,1} = -\beta_1 u_{n,1}. \end{aligned} \tag{1.14}$$

Inductively, we see that $\{u_{i,j+1}\}_{i=1}^{n+1}$ is uniquely determined by $\{u_{i,k}\}_{i=0}^{n+1}$, $k \leq j$.

We will introduce some initial boundary value problems of nonlinear partial difference equations in the later chapters.

1.3. The z-transform

Let $\{A_{m,n}\}$ be a double sequence, $(m,n) \in N_0^2$. The z-transform of this sequence is denoted by $\mathbf{Z}(A_{m,n})$ and is defined by

$$\mathbf{Z}(A_{m,n}) = F(z_1, z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n} z_1^{-m} z_2^{-n} \tag{1.15}$$

if series (1.15) converges for $|z_i| > r_i$, $r_i \geq 0$, $i = 1,2$. The notation $\mathbf{Z}$ denotes the operation of applying the z-transform. z_1 and z_2 are complex variables which may take any value in the complex plane. Equation (1.15) defines a complex analytic function of the variables z_1 and z_2 in the region $|z_1| > r_1$ and $|z_2| > r_2$.

Lemma 1.4. *Assume that there exist positive constants M_1, M, and N such that*

$$|A_{m,n}| \leq M_1 r_1^m r_2^n, \quad m \geq M,\ n \geq N. \tag{1.16}$$

Then the z-transform of $\{A_{m,n}\}$ exists in the region $|z_1| > r_1$ and $|z_2| > r_2$.

In the following, we assume that $A_{m,n} = 0$ for $m < 0$ and $n < 0$ in the series $\sum_{m=p}^{\infty}\sum_{n=q}^{\infty} A_{m,n} z_1^{-m} z_2^{-n}$. By direct calculations we can prove the following lemma easily.

Lemma 1.5. *The following formulas are true:*

(i)

$$\mathbf{Z}(A_{m-k,n-l}) = z_1^{-k} z_2^{-l} F(z_1, z_2); \tag{1.17}$$

(ii)

$$\sum_{i=0}^{\infty} F(k+i, z_2) z_1^{-i} = z_1^k \left(F(z_1, z_2) - \sum_{m=0}^{k-1} F(m, z_2) z_1^{-m} \right), \tag{1.18}$$

where $F(k+i, z_2) = \sum_{n=0}^{\infty} A_{k+i,n} z_2^{-n}$;

(iii)

$$\sum_{i=0}^{\infty} \sum_{n=0}^{l-1} A_{k+i,n} z_1^{-i} z_2^{-n} = z_1^k \left(\sum_{m=0}^{\infty} \sum_{n=0}^{l-1} A_{m,n} z_1^{-m} z_2^{-n} - \sum_{m=0}^{k-1} \sum_{n=0}^{l-1} A_{m,n} z_1^{-m} z_2^{-n} \right); \tag{1.19}$$

(iv)

$$\sum_{m=0}^{\infty} \sum_{n=0}^{l-1} A_{m,n} z_1^{-m} z_2^{-n} = \sum_{i=0}^{l-1} F(z_1, i) z_2^{-i}, \tag{1.20}$$

where $F(z_1, n) = \sum_{m=0}^{\infty} A_{m,n} z_1^{-m}$;

(v)

$$\begin{aligned} \mathbf{Z}(A_{m+k,n+l}) = z_1^k z_2^l \Bigg(& F(z_1, z_2) - \sum_{m=0}^{k-1} F(m, z_2) z_1^{-m} \\ & - \sum_{n=0}^{l-1} F(z_1, n) z_2^{-n} + \sum_{m=0}^{k-1} \sum_{n=0}^{l-1} A_{m,n} z_1^{-m} z_2^{-n} \Bigg). \end{aligned} \tag{1.21}$$

1.4. The Laplace transform

Assume that $A(x, y)$ is a real or complex value function of two real variables, defined on the region $D \equiv \{(x, y) \mid 0 \le x < \infty, 0 \le y < \infty\}$ and integrable in the Lebesgue sense over an arbitrary finite rectangle $D_{a,b}$ $(0 \le x \le a, 0 \le y \le b)$.

We will consider the expression

$$F(p, q; a, b) = \int_0^a \int_0^b e^{-px-qy} A(x, y) dx\, dy, \tag{1.22}$$

where $p = \sigma + i\mu$ and $q = \tau + iv$ are complex parameters determining a point (p, q) in the plane of two complex dimensions. Let S be the class of all functions $A(x, y)$ such that the following conditions are satisfied for at least one point (p, q).

(1) The integral (1.22) is bounded at the point (p,q) with respect to the variables a and b, that is,

$$|F(p,q;a,b)| < M(p,q) \quad \forall a \geq 0,\ b \geq 0, \tag{1.23}$$

where $M(p,q)$ is a positive constant independent of a and b.

(2) At the point (p,q),

$$\lim_{a,b\to\infty} F(p,q;a,b) = F(p,q) \tag{1.24}$$

exists. We denote the limit by

$$F(p,q) = L_{p,q}\{A(x,y)\} = \iint_0^\infty e^{-px-qy}A(x,y)dx\,dy. \tag{1.25}$$

The integral (1.25) is called the two-dimensional Laplace transform (or integral) of the function $A(x,y)$.

If the conditions 1 and 2 are satisfied simultaneously, we will say that the integral (1.25) converges boundedly for at least one point (p,q). Thus the class S consists of functions for which the integral (1.25) converges boundedly for at least one point (p,q). When the integral (1.25) converges boundedly, we will call $A(x,y)$ the determining function and $F(p,q)$ the generating function.

Remark 1.6. If the function $A(x,y)$ satisfies the condition

$$|A(x,y)| \leq Me^{hx+ky} \tag{1.26}$$

for all $x \geq 0$, $y \geq 0$, where M, h, k are positive constants, then it is easy to prove that $A(x,y)$ belongs to the class S at all points (p,q) for which $\operatorname{Re} p > h$, $\operatorname{Re} q > k$.

Theorem 1.7. *If the integral* (1.25) *converges boundedly at the point* (p_0,q_0), *then it converges boundedly at all points* (p,q) *for which* $\operatorname{Re}(p-p_0) > 0$, $\operatorname{Re}(q-q_0) > 0$.

1.5. Some useful results from functional analysis and function theory

Theorem 1.8 (Fabry theorem). *Let* $F(z_1,z_2)$ *be defined by*

$$F(z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}z_1^{-m}z_2^{-n}, \tag{1.27}$$

where z_1 *and* z_2 *are complex and* $|z_i| < a_i$, $i = 1,2$. *Assume that* $a_{m,n} = 1 + o(1/M)$, $M = \max(m,n)$. *Then* $F(z_1,z_2)$ *is singular at* $z_1 = 1$ *and* $z_2 = 1$.

Let Ω be a convex subset of R, and let $f : \Omega \to R$ be convex, that is,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \quad (x,y) \in \Omega,\ \alpha \in (0,1). \tag{1.28}$$

Then the following Jensen's inequality holds:

$$\frac{1}{b-a}\int_a^b f(x(t))dt \geq f\left(\frac{1}{b-a}\int_a^b x(t)dt\right). \tag{1.29}$$

A nonempty and closed subset E of a Banach space X is called a cone if it possesses the following properties.

(1) If $\alpha \in R^+$ and $\mu \in E$, then $\alpha\mu \in E$.

(2) If $\mu, \nu \in E$, then $\mu + \nu \in E$.

(3) If $\mu \in E - \{0\}$, then $-\mu \notin E$.

A Banach space X is partially ordered if it contains a cone E with nonempty interior. The ordering $\leq$ in X is defined by

$$u \leq v \quad \text{iff } v - u \in E. \tag{1.30}$$

Let S be a subset of a partially ordered Banach space X. Set

$$\overline{S} = \{u \in X : v \leq u \text{ for every } v \in S\}. \tag{1.31}$$

The point $u_0 \in X$ is the supremum of S if $u_0 \in S$ and for every $u \in \overline{S}$, $u \leq u_0$. Then infimum of S is defined in a similar way.

Theorem 1.9 (Knaster-Tarski fixed point theorem). *Let X be a partially ordered Banach space with ordering $\leq$. Let S be a subset of X with the property that the infimum of S belongs to S and every nonempty subset of S has a supremum which belongs to S. Let $T : S \to S$ be an increasing mapping, that is, $u \leq v$ implies that $Tu \leq Tv$. Then T has a fixed point in S.*

Remark 1.10. In Knaster-Tarski fixed point theorem the continuity of T is not required.

Theorem 1.11 (Brouwer fixed point theorem). *Let Ω be a nonempty, closed, bounded, and convex subset of R^n, and let $T : \Omega \to \Omega$ be a continuous mapping. Then T has a fixed point in Ω.*

Theorem 1.12 (Banach fixed point theorem). *Let (X, d) be a nonempty complete metric space and let $T : X \to X$ be a contraction mapping. Then T has a fixed point in X.*

Theorem 1.13 (Schauder fixed point theorem). *Let Ω be a nonempty, closed, and convex subset of a Banach space X. Let $T : \Omega \to \Omega$ be a continuous mapping such that $T\Omega$ is a relatively compact subset of X. Then T has at least one fixed point in Ω.*

Theorem 1.14 (Krasnoselskii fixed point theorem). *Let X be a Banach space and let Ω be a bounded closed convex subset of X. T_1 and T_2 are maps of Ω into X such that*

$T_1x + T_2y \in \Omega$ for every pair $x, y \in \Omega$. If T_1 is a contraction and T_2 is completely continuous, then the equation

$$T_1x + T_2x = x \tag{1.32}$$

has a solution in Ω.

1.6. Notes

Elementary discussions of partial difference equations and various applications are included in several books, for example, Levy and Lessman [89], Cheng [29], Kelley and Peterson [77], Agarwal [2], and so forth, also see [19, 22, 48, 49, 92, 118, 132]. In [2, 77, 89], authors only discuss the partial difference equations without delay. There are few discussions of the delay partial difference equations in [29]. The theory of delay partial differential equations can be found by Wu [154]. The behavior of differential equations can be different with its corresponding difference versions, see Hooker [70].

Example 1.1 is taken from Zhang and Saker [177]. Example 1.2 is taken from Cheng [29]. The initial value problem of (1.7) is posed by Zhang et al. [176] and Zhang and Liu [169]. Example 1.3 is taken from Cheng and Zhang [42]. Equation (1.10) is studied by Baĭnov and Mishev [15]. Theory of the z-transform can be seen from Vích [146], also see Gregor [67]. Laplace transform of two variables is taken from Ditkin and Prudnikov [54]. Theorem 1.8 is taken from Gilbert [63]. Some fixed point theorems in Section 1.5 are classical, which can be found in many books.

2 Oscillations of linear delay partial difference equations

2.1. Introduction

In this chapter, we will systematically describe the theory of oscillations of linear delay partial difference equations, that is, we study the existence and nonexistence of positive solutions of the initial value problem of linear delay partial difference equations. We will begin with linear PDEs with constant parameters by the analysis of characteristic equations and then discuss the case with variable coefficients presenting various available techniques. We present results for the equation with integer variables first, then we show which technique is needed for the equation with continuous arguments to the similar results.

2.2. Linear PDEs with constant parameters

Consider the delay partial difference equation

$$A_{m+1,n} + A_{m,n+1} - pA_{m,n} + \sum_{i=1}^{u} q_i A_{m-k_i,n-l_i} = 0, \quad m,n = 0,1,2,\ldots, \tag{2.1}$$

where p and q_i are real numbers, k_i and $l_i \in N_0$, $i = 1,2,\ldots,u$, u is a positive integer. A solution of (2.1) is a real double sequence $\{A_{i,j}\}$, $(i,j) \in N_0 \times N_1$, which satisfies (2.1).

A solution $\{A_{i,j}\}$ of (2.1) is said to be eventually positive (negative) if $A_{i,j} > 0$ ($A_{i,j} < 0$) for all large i and j. It is said to be oscillatory if it is neither eventually positive nor eventually negative. The purpose of this section is to derive a sufficient and necessary condition for all solutions of (2.1) to be oscillatory.

A solution $\{A_{i,j}\}$ of (2.1) is called to be proper if there exist positive numbers M, α, and β such that

$$|A_{m,n}| \le M\alpha^m\beta^n \tag{2.2}$$

for all large m and n.

It is not difficult to prove that if the initial data satisfy

$$|\phi_{m,n}| \leq M_1 \alpha^m \beta^n, \quad (m,n) \in \Omega, \tag{2.3}$$

for some positive numbers M_1, α, and β, then the corresponding solution is proper.

We look for the solution of the form

$$A_{m,n} = \lambda^m \mu^n, \tag{2.4}$$

where λ and μ are complex. Substituting (2.4) into (2.1), we obtain the characteristic equation

$$\Phi(\lambda,\mu) = \lambda + \mu - p + \sum_{i=1}^{u} q_i \lambda^{-k_i} \mu^{-l_i} = 0. \tag{2.5}$$

Theorem 2.1. *Every proper solution* $\{A_{m,n}\}$ *of* (2.1) *is oscillatory if and only if its characteristic equation* (2.5) *has no positive roots.*

Proof.

Necessity. Otherwise, let (λ_0, μ_0) be a positive root of (2.5). Then it is easy to find that $\{A_{m,n}\}$ with $A_{m,n} = \lambda_0^m \mu_0^n$ is a positive proper solution of (2.1), a contradiction.

Sufficiency. Assume that (2.5) has no positive roots. Let $\{A_{m,n}\}$ be a positive proper solution of (2.1) with the initial data $\phi_{m,n}$ such that $|\phi_{m,n}| < c$, $(m,n) \in \Omega$. Then, by induction, it is easy to find that there exists $b > 0$ such that

$$|A_{m,n}| < bc^{m+n}, \quad (m,n) \in N_0^2. \tag{2.6}$$

Hence, by Lemma 1.4, for $|z_i| > c$, $i = 1,2$, the z-transform of $\{A_{m,n}\}$

$$\mathbf{Z}(A_{m,n}) = \sum_{m,n=0}^{\infty} A_{m,n} z_1^{-m} z_2^{-n} = F(z_1,z_2) \tag{2.7}$$

exists. By taking the z-transform on both sides of (2.1), we obtain

$$\mathbf{Z}(A_{m+1,n}) + \mathbf{Z}(A_{m,n+1}) - p\mathbf{Z}(A_{m,n}) + \sum_{i=1}^{u} q_i \mathbf{Z}(A_{m-k_i,\,n-l_i}) = 0. \tag{2.8}$$

By Lemma 1.5, (2.8) becomes

$$\begin{aligned} &z_1 F(z_1,z_2) + z_2 F(z_1,z_2) - pF(z_1,z_2) + \sum_{i=1}^{u} q_i z_1^{-k_i} z_2^{-l_i} F(z_1,z_2) \\ &\quad - z_1 \sum_{n=0}^{\infty} A_{0,n} z_2^{-n} - z_2 \sum_{m=0}^{\infty} A_{m,0} z_1^{-m} = 0, \quad |z_i| > c,\ i = 1,2. \end{aligned} \tag{2.9}$$

Set

$$\begin{aligned}\Phi(z_1,z_2) &= z_1 + z_2 - p + \sum_{i=1}^{u} q_i z_1^{-k_i} z_2^{-l_i},\\ \psi(z_1,z_2) &= z_1 \sum_{n=0}^{\infty} A_{0,n} z_2^{-n} + z_2 \sum_{m=0}^{\infty} A_{m,0} z_1^{-m}.\end{aligned} \tag{2.10}$$

Then (2.9) becomes

$$\Phi(z_1,z_2)F(z_1,z_2) = \psi(z_1,z_2), \quad |z_i| > c,\ i = 1,2. \tag{2.11}$$

We rewrite (2.11) in the form

$$\Phi\left(\frac{1}{z_1},\frac{1}{z_2}\right)F\left(\frac{1}{z_1},\frac{1}{z_2}\right) = \psi\left(\frac{1}{z_1},\frac{1}{z_2}\right). \tag{2.12}$$

Set

$$w(z_1,z_2) = F\left(\frac{1}{z_1},\frac{1}{z_2}\right) = \sum_{m,n=0}^{\infty} A_{m,n} z_1^m z_2^n. \tag{2.13}$$

Equation (2.13) has the radius of convergence r_i, $i = 1,2$, that is, (2.12) holds for $|z_i| < r_i$, $i = 1,2$. Equivalently, (2.11) holds for $|z_i| > 1/r_i$, $i = 1,2$. By Theorem 1.8, a power series with positive coefficients having the radius of convergence r_i, $i = 1,2$ has the singularity at $z_i = r_i$, $i = 1,2$. Since (2.5) has no positive roots, we have $\Phi(z_1,z_2) \neq 0$ for $(z_1,z_2) \in (0,\infty)\times(0,\infty)$. Thus $\Phi(1/r_1,1/r_2) \neq 0$, and hence

$$w(z_1,z_2) = \frac{\psi(1/z_1,1/z_2)}{\Phi(1/z_1,1/z_2)} \tag{2.14}$$

is analytic in the region $|z_1 - r_1| < \rho_1$ and $|z_2 - r_2| < \rho_2$, where ρ_1 and ρ_2 are positive constants, which contradicts the singularity of $w(z_1,z_2)$ at $z_i = r_i$, $i = 1,2$. Therefore we must have $r_i = \infty$, $i = 1,2$, that is, (2.11) holds for $|z_i| > 0$, $i = 1,2$, which leads to $A_{m,n} = 0$ for all large m and n. Otherwise, the equality in (2.11) does not hold. This contradiction proves Theorem 2.1. □

From Theorem 2.1, we can derive an explicit condition for the oscillation of all proper solutions of (2.1).

Theorem 2.2. *Assume that* $p > 0$, $q_i \geq 0$, $i = 1,2,\dots,u$. *Then every proper solution of* (2.1) *oscillates if*

$$\sum_{i=1}^{u} q_i \frac{(k_i + l_i + 1)^{k_i+l_i+1}}{k_i^{k_i} l_i^{l_i} p^{k_i+l_i+1}} > 1. \tag{2.15}$$

Proof. If (2.15) holds, we are going to prove that the characteristic equation (2.5) has no positive roots. Clearly, (2.5) has no positive roots for $\lambda + \mu - p \geq 0$. For $\lambda + \mu - p < 0$, we write $\Phi(\lambda, \mu)$ in the form

$$\Phi(\lambda,\mu) = (p-\lambda-\mu)\left(-1+\sum_{i=1}^{u} q_i \frac{\lambda^{-k_i}\mu^{-l_i}}{p-\lambda-\mu}\right). \tag{2.16}$$

Set

$$f_i(\lambda,\mu) = \frac{\lambda^{-k_i}\mu^{-l_i}}{p-\lambda-\mu}. \tag{2.17}$$

Solving $\partial f_i/\partial\lambda = 0$ and $\partial f_i/\partial\mu = 0$, we obtain

$$\lambda_0 = \frac{pk_i}{k_i+l_i+1} > 0, \qquad \mu_0 = \frac{pl_i}{k_i+l_i+1} > 0. \tag{2.18}$$

It is easy to find that $f_i(\lambda,\mu)$ reaches its minimum value at (λ_0, μ_0), that is,

$$\min_{0<\lambda+\mu<p} f_i(\lambda,\mu) = f_i(\lambda_0,\mu_0) = \frac{(k_i+l_i+1)^{k_i+l_i+1}}{k_i^{k_i} l_i^{l_i} p^{k_i+l_i+1}}. \tag{2.19}$$

Hence, for $0 < \lambda + \mu < p$, we have

$$\Phi(\lambda,\mu) \geq (p-\lambda-\mu)\left(-1+\sum_{i=1}^{u} q_i \frac{(k_i+l_i+1)^{k_i+l_i+1}}{k_i^{k_i} l_i^{l_i} p^{k_i+l_i+1}}\right) > 0, \tag{2.20}$$

which implies that (2.5) has no positive roots. By Theorem 2.1, every proper solution of (2.1) oscillates. □

For u=1, (2.15) is not only sufficient but also necessary for every proper solution of (2.1) to be oscillatory.

Consider the equation

$$A_{m+1,n} + A_{m,n+1} - pA_{m,n} + qA_{m-k,\,n-l} = 0, \tag{2.21}$$

where $k, l \in N_0$.

Theorem 2.3. *Every proper solution of* (2.21) *oscillates if and only if*

$$q\frac{(k+l+1)^{k+l+1}}{k^k l^l p^{k+l+1}} > 1. \tag{2.22}$$

Proof. To prove this theorem it is sufficient to prove that if (2.22) does not hold, then (2.21) has a positive proper solution. In fact, the characteristic equation of (2.21) is

$$\Phi(\lambda,\mu) = \lambda + \mu - p + q\lambda^{-k}\mu^{-l} = 0. \tag{2.23}$$

Obviously, if (2.22) does not hold, then

$$\begin{gathered}\Phi\left(\frac{p(k+1)}{k+l+1}, \frac{pl}{k+l+1}\right) > 0,\\ \Phi\left(\frac{pk}{k+l+1}, \frac{pl}{k+l+1}\right) = \frac{p}{k+l+1}\left(-1 + q\frac{(k+l+1)^{k+l+1}}{k^k l^l p^{k+l+1}}\right) \le 0.\end{gathered} \tag{2.24}$$

Since $\Phi(\lambda,\mu)$ is continuous, then there exist

$$\lambda_0 \in \left[\frac{pk}{k+l+1}, \frac{p(k+1)}{k+l+1}\right), \qquad \mu_0 = \frac{pl}{k+l+1} \tag{2.25}$$

such that $\Phi(\lambda_0,\mu_0) = 0$. By Theorem 2.1, (2.21) has a positive solution. The proof is complete. □

The above method is available for other linear PDEs with constant parameters.

For example, we consider the hyperbolic type partial difference equation

$$A_{m-1,n} - A_{m,n-1} - pA_{m,n} + \sum_{i=1}^{u} q_i A_{m+k_i,\,n+l_i} = 0, \quad m,n = 0,1,2,\ldots, \tag{2.26}$$

where p, q_i are real numbers, k_i and $l_i \in N_0$, $i = 1,2,\ldots,u$, u is a positive integer. A solution of (2.26) is a real double sequence $\{A_{i,j}\}$, $(i,j) \in N_0^2$, which satisfies (2.26).

We look for the solution of the form (2.4). Substituting (2.4) into (2.26) we obtain the characteristic equation

$$\Phi(\lambda,\mu) = \lambda^{-1} - \mu^{-1} - p + \sum_{i=1}^{u} q_i\lambda^{k_i}\mu^{l_i} = 0. \tag{2.27}$$

Theorem 2.4. *Every proper solution* $\{A_{m,n}\}$ *of* (2.26) *is oscillatory if and only if its characteristic equation* (2.27) *has no positive roots.*

The proof is similar to the proof of Theorem 2.1.

From Theorem 2.4, we can obtain sufficient conditions, given explicitly in terms of the coefficients and the delays, for the oscillation of all proper solutions of (2.26).

Theorem 2.5. *Assume that* $p > 0$, $q_i \geq 0$, *and* $1 + k_i > l_i$, $i = 1, 2, \ldots, u$. *Then every proper solution of* (2.26) *oscillates if*

$$\sum_{i=1}^{u} q_i \frac{(k_i - l_i + 1)^{k_i - l_i + 1}}{k_i^{k_i} l_i^{l_i} p^{k_i + l_i + 1}} > 1. \tag{2.28}$$

For $u = 1$, (2.28) is not only sufficient but also necessary for the oscillation of all proper solutions of (2.26).

Consider the equation

$$A_{m-1,n} - A_{m,n-1} - pA_{m,n} + qA_{m+k,\,n+l} = 0, \quad m, n = 0, 1, 2, \ldots, \tag{2.29}$$

where $k, l \in N_0$.

Theorem 2.6. *Assume that* $p, q > 0$, $1 + k > l$. *Every proper solution of* (2.29) *oscillates if and only if*

$$q \frac{(k - l + 1)^{k-l+1}}{k^k l^l p^{k+l+1}} > 1. \tag{2.30}$$

Remark 2.7. From Theorem 2.1, the characteristic equation (2.1) plays an important role in the investigation of the oscillation of solutions of linear PDEs with constant parameters. But to determine if the characteristic equation has no positive roots is quite a problem itself. We want to find the necessary and sufficient condition expressed in terms p, q_i, k_i, l_i for the oscillation of (2.1), which is an open problem.

2.3. Systems of linear PDEs with constant parameters

Consider the linear partial difference system in the form

$$A_{m,n} = \sum_{i=1}^{u} p_i A_{m-k_i,\,n-l_i} + \sum_{j=1}^{v} q_j A_{m+\tau_j,\,n+\sigma_j}, \quad m, n = 0, 1, \ldots, \tag{2.31}$$

where p_i and q_j are $r \times r$ matrices, $A_{m,n} = (a^1_{m,n}, a^2_{m,n}, \ldots, a^r_{m,n})^T$, $k_i, l_i, \tau_j, \sigma_j \in N_0$, $i = 1, 2, \ldots, u$, $j = 1, 2, \ldots, v$, u and v are positive integers.

By a solution of (2.31) we mean a sequence $\{A_{m,n}\}$ of $A_{m,n} \in R^r$, which satisfies (2.31) for $m, n \in N_0$.

A sequence of real numbers $\{a^i_{m,n}\}$ is said to oscillate if the term $a^i_{m,n}$ is not all eventually positive or eventually negative in m, n. Let $\{A_{m,n}\}$ be a solution of (2.31) with $A_{m,n} = (a^1_{m,n}, a^2_{m,n}, \ldots, a^r_{m,n})^T$ for $m, n \in N_0$. We say that the solution $\{A_{m,n}\}$ oscillates componentwise if each component $\{a^i_{m,n}\}$ oscillates. Otherwise, the solution $\{A_{m,n}\}$ is called nonoscillatory. Therefore a solution of (2.31) is nonoscillatory if it has a component $\{a^i_{m,n}\}$, which is eventually positive or eventually negative in m, n.

The solution $\{A_{m,n}\}$ of (2.31) is said to be proper if there exist positive numbers M, α, and β such that

$$||A_{m,n}|| \le M\alpha^m\beta^n \quad \text{for } m,n \in N_0. \tag{2.32}$$

In the following, we will show the sufficient condition for all solutions to be proper for the linear difference system

$$A_{m,n} = \sum_{i=1}^{u} p_i A_{m-k_i,\,n-l_i}. \tag{2.33}$$

Set $k = \max k_i$, $l = \max l_i$, $i = 1,2,\ldots,u$, $\Omega = N_{-k} \times N_{-l} \setminus N_0 \times N_1$. Given a function $\phi_{i,j}$ defined on Ω, it is easy to construct by induction a double sequence $\{A_{i,j}\}$ which equals $\phi_{i,j}$ on Ω and satisfies (2.31) on $N_0 \times N_1$. It is not difficult to prove that if the initial data $\phi_{m,n}$ satisfy (2.32) on Ω, then the corresponding solution of (2.33) is proper. Similarly, we can find some conditions to guarantee that every solution of (2.31) is proper.

The purpose of this section is to derive the sufficient and necessary condition for all proper solutions of (2.31) to be oscillatory componentwise.

Theorem 2.8. *Every proper solution* $\{A_{m,n}\}$ *of* (2.31) *oscillates componentwise if and only if its characteristic equation*

$$\det\left(\sum_{i=1}^{u} p_i\lambda^{-k_i}\mu^{-l_i} - I + \sum_{j=1}^{v} q_j\lambda^{\tau_j}\mu^{\sigma_j}\right) = 0 \tag{2.34}$$

has no positive roots.

Proof. The proof of "only if" is simple. Suppose to the contrary, let (λ_0,μ_0) be a positive root of (2.34), then there would be a nonzero vector $\zeta \in R^r$ such that

$$\left(\sum_{i=1}^{u} p_i{\lambda_0}^{-k_i}{\mu_0}^{-l_i} - I + \sum_{j=1}^{v} q_j{\lambda_0}^{\tau_j}{\mu_0}^{\sigma_j}\right)\zeta = 0 \tag{2.35}$$

which implies that $A_{m,n} = \lambda_0^m\mu_0^n\zeta$ is a proper solution of (2.31) with at least one nonoscillatory component, which is a contradiction.

The proof of "if" uses the z-transform. Assume that (2.34) has no positive roots and (2.31) has a proper solution $\{A_{m,n}\}$ with at least one nonoscillatory component. Without loss of generality, we assume that $\{a^1_{m,n}\}$ is eventually positive. As (2.31) is autonomous, we may assume that $a^1_{m,n} > 0$ for $m,n \in N_0$. For the proper solution $\{A_{m,n}\}$, the z-transform

$$\mathbf{Z}(A_{m,n}) = \sum_{m,n=0}^{\infty} A_{m,n}z_1^{-m}z_2^{-n} = F(z_1,z_2) \tag{2.36}$$

exists for $|z_1| > \alpha > 0$, $|z_2| > \beta > 0$. By taking the z-transform of both sides of (2.31) and using some formulas of the z-transform in Section 1.3, we obtain

$$\phi(z_1, z_2)F(z_1, z_2) = \psi(z_1, z_2), \tag{2.37}$$

where

$$\begin{aligned} \phi(z_1, z_2) &= \sum_{i=1}^{u} p_i z_1^{-k_i} z_2^{-l_i} + \sum_{j=1}^{v} q_j z_1^{\tau_j} z_2^{\sigma_j} - I, \\ \psi(z_1, z_2) &= \sum_{j=1}^{v} q_j z_1^{\tau_j} z_2^{\sigma_j} \Bigg(\sum_{m=0}^{\tau_j - 1} \sum_{n=0}^{\infty} A_{m,n} z_1^{-m} z_2^{-n} + \sum_{n=0}^{\sigma_j - 1} \sum_{m=0}^{\infty} A_{m,n} z_1^{-m} z_2^{-n} \\ &\qquad - \sum_{m=0}^{\tau_j - 1} \sum_{n=0}^{\sigma_j - 1} A_{m,n} z_1^{-m} z_2^{-n} \Bigg). \end{aligned} \tag{2.38}$$

By condition (2.34), $\det \phi(z_1, z_2) \neq 0$ for $z_1 \times z_2 \in (0, \infty)^2$. Let $F_1(z_1, z_2)$ be the z-transform of the first component $\{a_{m,n}^1\}$ of the solution $\{A_{m,n}\}$ and let b be the modulus of the largest zero of $\det \phi(z_1, z_2)$. Then by Cramer's rule, for $|z_1| > \max\{\alpha, b\}$, $|z_2| > \max\{\beta, b\}$,

$$\det \phi(z_1, z_2)F_1(z_1, z_2) = \det D(z_1, z_2), \tag{2.39}$$

where $D(z_1, z_2)$ has components of $\phi(z_1, z_2)$ and $\psi(z_1, z_2)$ as its entries and

$$F_1(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}^1 z_1^{-m} z_2^{-n}. \tag{2.40}$$

Let

$$w_1(z_1, z_2) = F_1\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = \sum_{m,n=0}^{\infty} a_{m,n}^1 z_1^m z_2^n. \tag{2.41}$$

Equation (2.41) is a power series with positive coefficients having the radius of convergence ρ_i, $i = 1, 2$. Hence

$$\det \phi\left(\frac{1}{z_1}, \frac{1}{z_2}\right) w_1(z_1, z_2) = \det D\left(\frac{1}{z_1}, \frac{1}{z_2}\right), \tag{2.42}$$

for $|z_i| < \rho_i$, $i = 1, 2$. By Theorem 1.8, a power series with positive coefficients having the radius of convergence ρ_i, $i = 1, 2$ has the singularity at $z_i = \rho_i$, $i = 1, 2$. By condition $\det \phi(z_1, z_2) \neq 0$ for $(z_1, z_2) \in (0, \infty) \times (0, \infty)$. Thus $\det \phi(1/\rho_1, 1/\rho_2) \neq 0$, and hence

$$w_1(z_1, z_2) = \frac{\det D(1/z_1, 1/z_2)}{\det \phi(1/z_1, 1/z_2)} \tag{2.43}$$

is analytic in the regions $|z_1 - \rho_1| < d_1$ and $|z_2 - \rho_2| < d_2$, where d_1 and d_2 are positive constants, which contradicts the singularity of $w_1(z_1, z_2)$ at $z_i = \rho_i$, $i = 1, 2$. Therefore, we must have $\rho_i = \infty$, $i = 1, 2$, that is, (2.39) holds for $|z_i| > 0$, $i = 1, 2$, which leads to $a^1_{m,n} = 0$ for all large m and n. Otherwise, for any fixed large numbers M and N, the left-hand side of (2.39) has the nonzero term $b_{m,n} z_1^{-m} z_2^{-n}$, where $m \geq M$ and $n \geq N$. But the right-hand side of (2.39) has no such term. This contradiction proves Theorem 2.8. □

For the scalar linear difference equation

$$a_{m,n} = \sum_{i=1}^{u} p_i a_{m-k_i, n-l_i} + \sum_{j=1}^{v} q_j a_{m+\tau_j, n+\sigma_j}, \quad m, n = 0, 1, 2, \ldots, \tag{2.44}$$

we have the following result.

Corollary 2.9. *Every proper solution of* (2.44) *oscillates if and only if the characteristic equation*

$$1 = \sum_{i=1}^{u} p_i \lambda^{-k_i} \mu^{-l_i} + \sum_{j=1}^{v} q_j \lambda^{\tau_j} \mu^{\sigma_j} \tag{2.45}$$

has no positive roots.

From Corollary 2.9 we can derive explicit conditions for the oscillation of all proper solutions of some special equations.

2.4. Linear PDEs with continuous arguments

In this section, we will consider the linear delay partial difference equation with continuous arguments

$$A(x+1, y) + A(x, y+1) - A(x, y) + pA(x - \sigma, y - \tau) = 0, \tag{2.46}$$

where $p \in R$, $\tau \geq 0$, $\sigma \geq 0$.

By a solution of (2.46) we mean a continuous function $A \in C([-\sigma, \infty) \times [-\tau, \infty), R)$, which satisfies (2.46) for all $x \geq 1$, $y \geq 1$. Let $\Omega = [-\sigma, +\infty) \times [-\tau, +\infty) \setminus [1, +\infty) \times [1, +\infty)$. Given an initial function $\phi(x, y) \in C(\Omega, R)$, by the method of steps, one can see that (2.46) has a unique solution on $[1, \infty) \times [1, \infty)$, which satisfies the initial condition on Ω.

A solution $A(x, y)$ of (2.46) is said to be eventually positive (negative) if $A(x, y) > 0$ ($A(x, y) < 0$) for all large x and y. It is said to be oscillatory if it is neither eventually positive nor eventually negative.

A solution $A(x, y)$ is said to be proper, if there are positive constants M, h, and k such that

$$|A(x, y)| \leq Me^{hx+ky} \quad \text{for all sufficiently large } x \text{ and } y. \tag{2.47}$$

It is easy to prove that if the initial function $\phi(x,y)$ satisfies $|\phi(x,y)| \le M\exp(h_1x+k_1y)$, $h_1 > 0$, $k_1 > 0$, $(x,y) \in \Omega$, then the corresponding solution of (2.46) is proper.

Consider (2.46) together with its characteristic equation

$$\phi(\lambda,\mu) = \lambda + \mu - 1 + p\lambda^{-\sigma}\mu^{-\tau} = 0. \tag{2.48}$$

Theorem 2.10. *Every proper solution $A(x,y)$ of (2.46) is oscillatory if and only if the characteristic equation (2.48) has no positive roots.*

Proof.
Necessity. Otherwise, let (λ_0,μ_0) be a positive root of (2.48). Then it is easy to find that $A(x,y) = \lambda_0^x\mu_0^y$ is a proper positive solution of (2.46), a contradiction.

Sufficiency. Assume that (2.48) has no positive roots. Let $A(x,y)$ be a proper positive solution of (2.46). By Theorem 1.7, for $\operatorname{Re} s > h$, $\operatorname{Re} q > k$, the Laplace transform of $A(x,y)$

$$F(s,q) = L_{s,q}\{A(x,y)\} = \iint_0^\infty e^{-sx-qy}A(x,y)\,dx\,dy \tag{2.49}$$

exists. Taking the Laplace transform on both sides of (2.46), we obtain

$$f(s,q)F(s,q) = W(s,q), \tag{2.50}$$

where

$$\begin{aligned}
f(s,q) &= e^s + e^q - 1 + pe^{-s\sigma-q\tau},\\
W(s,q) &= e^s\int_0^{+\infty}\int_0^1 e^{-sx-qy}\phi(x,y)\,dx\,dy\\
&\qquad + e^q\int_0^{+\infty}\int_0^1 e^{-sx-qy}\phi(x,y)\,dy\,dx + \psi(s,q),\\
\psi(s,q) &= -pe^{-s\sigma-q\tau}\Big(\int_0^{+\infty}\int_{-\tau}^0 e^{-sx-qy}\phi(x,y)\,dx\,dy\\
&\qquad + \int_0^{+\infty}\int_{-\sigma}^0 e^{-sx-qy}\phi(x,y)\,dy\,dx\\
&\qquad + \int_{-\sigma}^0\int_{-\tau}^0 e^{-sx-qy}\phi(x,y)\,dx\,dy\Big).
\end{aligned} \tag{2.51}$$

Since $F(s,q)$ is the Laplace transform of a positive function, if $s_0 > -\infty$, $q_0 > -\infty$, in the sense of analytic continuous, $F(s,q)$ must have the singularity at point (s_0,q_0). But $W(s,q)$ is an entire function of (s,q) on the two-dimensional plane, and because $f(s,q) = \Phi(e^s,e^q)$, so $f(s,q) \ne 0$ for all real (s,q). Equation (2.50) shows that $F(s,q)$ can be analytically continued to a neighborhood of any real (s,q). Thus, we must have $s_0 = -\infty$, $q_0 = -\infty$, and (2.50) holds for all real (s,q). Now $\lim_{s,q\to+\infty} f(s,q) = +\infty$, so $f(s,q) > 0$ for all (s,q). $f(s,q)$ is dominated by $pe^{-s\sigma-q\tau}$ as $s \to -\infty$, $q \to -\infty$, so we must have $p > 0$. On the other hand, $W(s,q)$ is dominated by ψ, as $s \to -\infty$, $q \to -\infty$. Since $p > 0$, $e^{-s\sigma-q\tau} > 0$, and $\phi(x,y) > 0$, we conclude that $W(s,q)$ is negative as $s \to -\infty$, $q \to -\infty$. But $F(s,q) \geq 0$. It is a contradiction. This contradiction proves Theorem 2.10. □

The next result provides explicit conditions for the oscillation of all proper solutions of (2.46).

Theorem 2.11. *Assume that* $p > 0$. *Then every proper solution of* (2.46) *oscillates if and only if*

$$p\frac{(\sigma+\tau+1)^{\sigma+\tau+1}}{\sigma^\sigma\tau^\tau} > 1. \tag{2.52}$$

Proof. To prove the necessity of this theorem, we need to prove that if (2.52) does not hold, then (2.46) has a positive proper solution. In fact, the characteristic equation of (2.46) is

$$\Phi(\lambda,\mu) = \lambda+\mu-1+p\lambda^{-\sigma}\mu^{-\tau} = 0. \tag{2.53}$$

Obviously,

$$\begin{gathered}\Phi\left(\frac{\sigma+1}{\sigma+\tau+1},\frac{\tau}{\sigma+\tau+1}\right) > 0,\\ \Phi\left(\frac{\sigma}{\sigma+\tau+1},\frac{\tau}{\sigma+\tau+1}\right) = \frac{1}{\sigma+\tau+1}\left(-1+p\frac{(\sigma+\tau+1)^{\sigma+\tau+1}}{\sigma^\sigma\tau^\tau}\right) \leq 0.\end{gathered} \tag{2.54}$$

Since Φ is continuous, there exist

$$\lambda_0 \in \left[\frac{\sigma}{\sigma+\tau+1},\frac{\sigma+1}{\sigma+\tau+1}\right), \qquad \mu_0 = \frac{\tau}{\sigma+\tau+1} \tag{2.55}$$

such that $\Phi(\lambda_0,\mu_0) = 0$. Then by Theorem 2.10, (2.46) has a positive proper solution.

Sufficiency. If (2.52) holds, we are going to prove that the characteristic equation (2.48) has no positive roots. Clearly, (2.48) has no positive roots for $\lambda+\mu \geq 1$. For $\lambda+\mu < 1$, we have

$$\Phi(\lambda,\mu) = (1-\lambda-\mu)\left(-1+\frac{p\lambda^{-\sigma}\mu^{-\tau}}{1-\lambda-\mu}\right). \tag{2.56}$$

Set

$$f(\lambda,\mu) = \frac{\lambda^{-\sigma}\mu^{-\tau}}{1-\lambda-\mu}. \tag{2.57}$$

It is easy to see that $f(\lambda,\mu)$ reaches its minimum value at $\lambda_0 = \sigma/(\sigma+\tau+1)$, $\mu_0 = \tau/(\sigma+\tau+1)$, that is,

$$\min_{0<\lambda+\mu<1} f(\lambda,\mu) = \frac{(\sigma+\tau+1)^{\sigma+\tau+1}}{\sigma^\sigma\tau^\tau}. \tag{2.58}$$

Hence, for $0<\lambda+\mu<1$, we have

$$\Phi(\lambda,\mu) \geq (1-\lambda-\mu)\left(-1+\frac{p(\sigma+\tau+1)^{\sigma+\tau+1}}{\sigma^\sigma\tau^\tau}\right) > 0 \tag{2.59}$$

which implies that (2.48) has no positive roots. By Theorem 2.10, the proof is completed. □

Example 2.12. Consider the partial difference equation

$$A(x+1,y)+A(x,y+1)-A(x,y)+A(x-2,y-4)=0, \tag{2.60}$$

its characteristic equation is

$$\lambda+\mu-1+\lambda^{-2}\mu^{-4}=0. \tag{2.61}$$

Obviously, (2.61) has no positive roots.

By Theorem 2.10, every proper solution of (2.60) is oscillatory. It is easy to find that $A(x,y) = c_1\sin\pi x + c_2\sin\pi y$ is a proper solution of (2.60) and is oscillatory, where c_1 and c_2 are arbitrary constants.

The above results can be extended to the partial difference equation with several delays of the form

$$A(x+1,y)+A(x,y+1)-pA(x,y)+\sum_{i=1}^{n} p_iA(x-\sigma_i,y-\tau_i)=0, \tag{2.62}$$

where $p>0$, $\sigma_i,\tau_i\in(0,\infty)$, $i=1,2,\dots,n$.

Theorem 2.13. *Every solution of* (2.62) *is oscillatory if and only if its characteristic equation*

$$\Phi(\lambda,\mu)=\lambda+\mu-p+\sum_{i=1}^{n} p_i\lambda^{-\sigma_i}\mu^{-\tau_i}=0 \tag{2.63}$$

has no positive roots.

Theorem 2.14. *Assume that* $p > 0$, $p_i > 0$, $i = 1, 2, \ldots, n$, *then every solution of* (2.62) *oscillates if*

$$\sum_{i=1}^{n} p_i \frac{(\sigma_i + \tau_i + 1)^{\sigma_i+\tau_i+1}}{\sigma_i^{\sigma_i} \tau_i^{\tau_i} p^{\sigma_i+\tau_i+1}} > 1. \tag{2.64}$$

2.5. Linear PDEs with variable coefficients

2.5.1. Oscillation of PDEs with variable coefficients (I)

Consider the linear delay partial difference equation

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + p_{m,n}A_{m-k,n-l} = 0, \tag{2.65}$$

where $p_{m,n} > 0$ on N_0^2, $k, l \in N_0$.

A double sequence $\{A_{m,n}\}$, $(m, n) \in N_{m_0} \times N_{n_0}$ is said to be a solution of (2.65), if it satisfies (2.65) for $m \geq m_0$, $n \geq n_0$.

We assume that a, b, c, d, and $p_{m,n}$, $(m, n) \in N_{m_0} \times N_{n_0}$ are positive.

Define the set E by

$$E = \{\lambda > 0 \mid d - \lambda p_{m,n} > 0 \text{ eventually}\}. \tag{2.66}$$

Theorem 2.15. *Assume that*

(i) $\limsup_{m,n\to\infty} p_{m,n} > 0$;

(ii) *for* $k \geq l \geq 1$, *there exist* $M, N \in N_1$ *such that*

$$\sup_{\lambda\in E,\, m\geq M,\, n\geq N} \lambda \prod_{i=1}^{l} (d - \lambda p_{m-i,n-i}) \prod_{j=1}^{k-l} (d - \lambda p_{m-l-j,n-l}) < \left(a + \frac{2bc}{d}\right)^{l} b^{k-l}, \tag{2.67}$$

and for $l \geq k \geq 1$,

$$\sup_{\lambda\in E,\, m\geq M,\, n\geq N} \lambda \prod_{i=1}^{k} (d - \lambda p_{m-i,n-i}) \prod_{j=1}^{l-k} (d - \lambda p_{m-k,n-k-j}) < \left(a + \frac{2bc}{d}\right)^{k} c^{l-k}. \tag{2.68}$$

Then every solution of (2.65) *oscillates, where* $\prod_{j=1}^{0} * = 1$.

Proof. Suppose to the contrary, let $\{A_{m,n}\}$ be an eventually positive solution of (2.65). We define the set $S(A)$ of positive numbers by

$$S(A) = \{\lambda > 0 \mid aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - (d - \lambda p_{m,n})A_{m,n} \leq 0 \text{ eventually}\}. \tag{2.69}$$

From (2.65), we have

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} < dA_{m,n}. \tag{2.70}$$

If $k \geq l$, then we obtain

$$A_{m-k,n-l} > \left(\frac{a}{d}\right)^{l} A_{m-k+l,n} > \left(\frac{a}{d}\right)^{l}\left(\frac{b}{d}\right)^{k-l} A_{m,n}. \tag{2.71}$$

If $l \geq k$, then we obtain

$$A_{m-k,n-l} > \left(\frac{a}{d}\right)^{k} A_{m,n-l+k} > \left(\frac{a}{d}\right)^{k}\left(\frac{c}{d}\right)^{l-k} A_{m,n}. \tag{2.72}$$

Substituting (2.71) and (2.72) into (2.65), we obtain

$$\begin{aligned}
&aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + \left(\frac{a}{d}\right)^{l}\left(\frac{b}{d}\right)^{k-l} p_{m,n}A_{m,n} < 0,\\
&aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + \left(\frac{a}{d}\right)^{k}\left(\frac{c}{d}\right)^{l-k} p_{m,n}A_{m,n} < 0,
\end{aligned} \tag{2.73}$$

respectively. Equations (2.73) show that $S(A)$ is nonempty. For $\lambda \in S(A)$, we have eventually

$$d - \lambda p_{m,n} > 0, \tag{2.74}$$

which implies that $S(A) \subseteq E$. Due to condition (i), the set E is bounded, and hence $S(A)$ is bounded. Let $\mu \in S(A)$. From (2.70), we have

$$A_{m+1,n+1} \leq \frac{d}{b} A_{m,n+1}, \qquad A_{m+1,n+1} \leq \frac{d}{c} A_{m+1,n}. \tag{2.75}$$

Hence, we obtain

$$\left(a + \frac{2bc}{d}\right) A_{m+1,n+1} \leq aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \leq (d - \mu p_{m,n}) A_{m,n}. \tag{2.76}$$

If $k \geq l$, we have

$$\begin{aligned}
A_{m,n} &\leq \left(a + \frac{2bc}{d}\right)^{-l} \prod_{i=1}^{l} (d - \mu p_{m-i,n-i}) A_{m-l,n-l},\\
A_{m-l,n-l} &\leq \frac{1}{b}(d - \mu p_{m-l-1,n-l}) A_{m-l-1,n-l}\\
&\leq \cdots \leq \left(\frac{1}{b}\right)^{k-l} \prod_{j=1}^{k-l} (d - \mu p_{m-l-j,n-l}) A_{m-k,n-l}.
\end{aligned} \tag{2.77}$$

Combining the above two inequalities, we obtain

$$A_{m,n} \leq \left(a+\frac{2bc}{d}\right)^{-l} b^{l-k} \prod_{i=1}^{l}\left(d-\mu p_{m-i,n-i}\right) \prod_{j=1}^{k-l}\left(d-\mu p_{m-l-j,n-l}\right) A_{m-k,n-l}. \tag{2.78}$$

Similarly, if $l \geq k$, we have

$$A_{m,n} \leq \left(a+\frac{2bc}{d}\right)^{-k} c^{k-l} \prod_{i=1}^{k}\left(d-\mu p_{m-i,n-i}\right) \prod_{j=1}^{l-k}\left(d-\mu p_{m-k,n-k-j}\right) A_{m-k,n-l}. \tag{2.79}$$

Substituting (2.78) and (2.79) into (2.65), we find, for $l \geq k$,

$$\begin{aligned} &A_{m+1,n+1}+bA_{m+1,n}+cA_{m,n+1}-dA_{m,n}+p_{m,n}\left(a+\frac{2bc}{d}\right)^{k} c^{l-k} \\ &\quad \times\left(\prod_{i=1}^{k}\left(d-\mu p_{m-i,n-i}\right) \prod_{j=1}^{l-k}\left(d-\mu p_{m-k,n-k-j}\right)\right)^{-1} A_{m,n} \leq 0, \end{aligned} \tag{2.80}$$

and, for $k \geq l$,

$$\begin{aligned} &aA_{m+1,n+1}+bA_{m+1,n}+cA_{m,n+1}-dA_{m,n}+p_{m,n}\left(a+\frac{2bc}{d}\right)^{l} b^{k-l} \\ &\quad \times\left(\prod_{i=1}^{l}\left(d-\mu p_{m-i,n-i}\right) \prod_{j=1}^{k-l}\left(d-\mu p_{m-l-j,n-l}\right)\right)^{-1} A_{m,n} \leq 0. \end{aligned} \tag{2.81}$$

Hence we have, for $l \geq k$,

$$\begin{aligned} &aA_{m+1,n+1}+bA_{m+1,n}+cA_{m,n+1} \\ &\quad -\Bigg(d-p_{m,n}\left(a+\frac{2bc}{d}\right)^{k} c^{l-k} \\ &\qquad \times \sup_{m\geq M,\, n\geq N}\left[\prod_{i=1}^{k}\left(d-\mu p_{m-i,n-i}\right) \prod_{j=1}^{l-k}\left(d-\mu p_{m-k,n-k-j}\right)\right]^{-1}\Bigg) A_{m,n} \leq 0, \end{aligned} \tag{2.82}$$

and, for $k \geq l$,

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - \left(d - p_{m,n}\left(a + \frac{2bc}{d}\right)^l b^{k-l} \times \sup_{m\geq M, n\geq N}\left[\prod_{i=1}^{l}(d-\mu p_{m-i,n-i})\prod_{j=1}^{k-l}(d-\mu p_{m-l-j,n-l})\right]^{-1}\right)A_{m,n} \leq 0. \tag{2.83}$$

From (2.82) and (2.83), we obtain, for $l \geq k$,

$$\left(a+\frac{2bc}{d}\right)^k c^{l-k}\times\left(\sup_{m\geq M, n\geq N}\left[\prod_{i=1}^{k}(d-\mu p_{m-i,n-i})\prod_{j=1}^{l-k}(d-\mu p_{m-k,n-k-j})\right]^{-1}\right)\in S(A), \tag{2.84}$$

and, for $k > l$,

$$\left(a+\frac{2bc}{d}\right)^l b^{k-l}\times\left(\sup_{m\geq M, n\geq N}\left[\prod_{i=1}^{l}(d-\mu p_{m-i,n-i})\prod_{j=1}^{k-l}(d-\mu p_{m-l-j,n-l})\right]^{-1}\right)\in S(A). \tag{2.85}$$

On the other hand, (2.67) implies that there exists $\beta \in (0,1)$ such that

$$\sup_{\lambda\in E, m\geq M, n\geq N}\lambda\prod_{i=1}^{l}(d-\lambda p_{m-i,n-i})\prod_{j=1}^{k-l}(d-\lambda p_{m-l-j,n-l})\leq\beta\left(a+\frac{2bc}{d}\right)^l b^{k-l},\quad k\geq l, \tag{2.86}$$

and (2.68) implies that there exists $\beta \in (0,1)$ such that

$$\sup_{\lambda\in E, m\geq M, n\geq N}\lambda\prod_{i=1}^{k}(d-\lambda p_{m-i,n-i})\prod_{j=1}^{l-k}(d-\lambda p_{m-k,n-k-j})\leq\beta\left(a+\frac{2bc}{d}\right)^k c^{l-k},\quad l\geq k. \tag{2.87}$$

Hence, for $k \geq l$, we have

$$\sup_{m\geq M, n\geq N}\left[\prod_{i=1}^{l}(d-\mu p_{m-i,n-i})\prod_{j=1}^{k-l}(d-\mu p_{m-l-j,n-l})\right]\leq\frac{\beta}{\mu}\left(a+\frac{2bc}{d}\right)^l b^{k-l}, \tag{2.88}$$

and, for $l \geq k$, we have

$$\sup_{m\geq M,\, n\geq N}\left[\prod_{i=1}^{k}(d-\mu p_{m-i,n-i})\prod_{j=1}^{l-k}(d-\mu p_{m-k,n-k-j})\right] \leq \frac{\beta}{\mu}\left(a+\frac{2bc}{d}\right)^{k}c^{l-k}. \tag{2.89}$$

From (2.84) and (2.89), for $l \geq k$, (2.85) and (2.88), for $k \geq l$, we have that $\mu/\beta \in S(A)$. Repeating the above procedure, we conclude that $\mu/\beta^r \in S(A)$, $r = 1, 2, \ldots$, which contradicts the boundedness of $S(A)$. The proof is complete. □

From Theorem 2.15, we can derive an explicit condition for the oscillation of (2.65).

Corollary 2.16. *In addition to (i) of Theorem 2.15, assume that*

$$\liminf_{m,n\to\infty} p_{m,n} = P > d^{k+1}\left(\left(a+\frac{2bc}{d}\right)^{l}b^{k-l}\right)^{-1}\frac{k^k}{(1+k)^{1+k}}, \quad k \geq l, \tag{2.90}$$

or

$$\liminf_{m,n\to\infty} p_{m,n} = P > d^{l+1}\left(\left(a+\frac{2bc}{d}\right)^{k}c^{l-k}\right)^{-1}\frac{l^l}{(1+l)^{1+l}}, \quad l \geq k. \tag{2.91}$$

Then every solution of (2.65) *oscillates.*

Proof. We see that

$$\max_{d/P>\lambda>0} \lambda(d-\lambda P)^k = \frac{d^{k+1}k^k}{P(1+k)^{1+k}}. \tag{2.92}$$

Hence (2.90) and (2.91) imply that (2.67) and (2.68) hold. By Theorem 2.15, every solution of (2.65) oscillates. The proof is complete. □

From (2.65), we have

$$A_{m,n} < \frac{d}{b}A_{m-1,n} < \cdots < \left(\frac{d}{b}\right)^{k}A_{m-k,n} < \cdots < \left(\frac{d}{b}\right)^{k}\left(\frac{d}{c}\right)^{l}A_{m-k,n-l}. \tag{2.93}$$

Let $\mu \in S(A)$. Then

$$\begin{aligned} A_{m,n} &\leq \frac{1}{b}(d-\mu p_{m-1,n})A_{m-1,n} \leq \left(\frac{1}{b}\right)^{k}\prod_{i=1}^{k}(d-\mu p_{m-i,n})A_{m-k,n} \\ &\leq \left(\frac{1}{b}\right)^{k}\left(\frac{1}{c}\right)\prod_{i=1}^{k}(d-\mu p_{m-i,n})(d-\mu p_{m-k,n-1})A_{m-k,n-1} \\ &\leq \left(\frac{1}{b}\right)^{k}\left(\frac{1}{c}\right)^{l}\prod_{i=1}^{k}(d-\mu p_{m-i,n})\prod_{j=1}^{l}(d-\mu p_{m-k,n-j})A_{m-k,n-l}. \end{aligned} \tag{2.94}$$

Substituting the above inequality into (2.65), we obtain

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + p_{m,n}b^k c^l \left[\prod_{i=1}^{k} (d - \mu p_{m-i,n}) \prod_{j=1}^{l} (d - \mu p_{m-k,n-j}) \right]^{-1} A_{m,n} \le 0. \tag{2.95}$$

Hence, we have

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - \left(d - p_{m,n}b^k c^l \left[\sup_{m\ge M,\, n\ge M} \prod_{i=1}^{k} (d - \mu p_{m-i,n}) \prod_{j=1}^{l} (d - \mu p_{m-k,n-j}) \right]^{-1} \right) A_{m,n} \le 0, \tag{2.96}$$

which implies that

$$b^k c^l \left[\sup_{m\ge M,\, n\ge N} \prod_{i=1}^{k} (d - \mu p_{m-i,n}) \prod_{j=1}^{l} (d - \mu p_{m-k,n-j}) \right]^{-1} \in S(A). \tag{2.97}$$

We are ready to state the following theorem.

Theorem 2.17. *In addition to (i) of Theorem 2.15, further, assume that*

(ii) *there exist $M, N \in N_1$ such that*

$$\sup_{\lambda\in E,\, m\ge M,\, n\ge N} \lambda \prod_{i=1}^{k} (d - \lambda p_{m-i,n}) \prod_{j=1}^{l} (d - \lambda p_{m-k,n-j}) < b^k c^l. \tag{2.98}$$

Then every solution of (2.65) *oscillates.*

Since

$$\max_{d/P>\lambda>0} \lambda (d - \lambda P)^{k+l} = \frac{d^{k+l+1}(k+l)^{k+l}}{P(1+k+l)^{1+k+l}} \tag{2.99}$$

and (2.98), we have the following result.

Corollary 2.18. *In addition to (i) of Theorem 2.15, assume that*

$$\liminf_{m,n\to\infty} p_{m,n} = P > \frac{d^{1+k+l}(k+l)^{k+l}}{b^k c^l (1+k+l)^{1+k+l}}. \tag{2.100}$$

Then every solution of (2.65) *is oscillatory.*

Example 2.19. Consider the partial difference equation

$$A_{m+1,n+1} + eA_{m+1,n} + A_{m,n+1} - A_{m,n} + (1+e)e^4 A_{m-2,n-2} = 0. \tag{2.101}$$

It is easy to see that (2.101) satisfies the conditions of Corollary 2.18, so every solution of this equation is oscillatory. In fact, $A_{m,n} = (-e)^{m+n}$ is such a solution.

Remark 2.20. Results in Section 2.5.1 are true for $a = 0$ in (2.65).

In the following we present the techniques to improve the results in Section 2.5.1.

2.5.2. Oscillation of PDEs with variable coefficients (II)

To obtain main results in this section, we need the following technical lemmas. The first lemma is obvious.

Lemma 2.21. *Assume that for positive integers $\overline{m}$, $\overline{n}$, and $r \geq 1$, (2.65) has a solution $\{A_{m,n}\}$ such that $A_{m,n} > 0$ for $m \in \{\overline{m}-k, \overline{m}-k+1, \dots, \overline{m}+r\}$ and $n \in \{\overline{n}-l, \overline{n}-l+1, \dots, \overline{n}+r\}$, and $p_{m,n} \geq 0$ for $m \in \{\overline{m}, \overline{m}+1, \dots, \overline{m}+r\}$ and $n \in \{\overline{n}, \overline{n}+1, \dots, \overline{n}+r\}$. Then*

$$d^r A_{\overline{m},\overline{n}} \geq a^r A_{\overline{m}+r,\overline{n}+r}, \qquad d^r A_{\overline{m},\overline{n}} \geq b^r A_{\overline{m}+r,\overline{n}}, \qquad d^r A_{\overline{m},\overline{n}} \geq c^r A_{\overline{m},\overline{n}+r}. \tag{2.102}$$

Lemma 2.22. *Let $r \geq 1$, $\overline{m}$ and $\overline{n}$ be positive integers so that $\overline{m} \geq 2k$ and $\overline{n} \geq 2l$. Assume that $\{A_{m,n}\}$ is a solution of (2.65) with $A_{m,n} > 0$ for $m \in \{\overline{m}-2k, \overline{m}-2k+1, \dots, \overline{m}+r\}$ and $n \in \{\overline{n}-2l, \overline{n}-2l+1, \dots, \overline{n}+r\}$ and $p_{m,n} \geq q \geq 0$ for $m \in \{\overline{m}-k, \overline{m}-k+1, \dots, \overline{m}+r\}$ and $n \in \{\overline{n}-l, \overline{n}-l+1, \dots, \overline{n}+r\}$. Then*

$$\begin{aligned} d^r A_{\overline{m},\overline{n}} \geq{} & a \sum_{j=0}^{r-1} d^{r-1-j} \sum_{i=0}^{j} b^{j-i} c^i C_j^i A_{\overline{m}+j+1-i,\overline{n}+1+i} \\ & + aq \sum_{j=0}^{r-2} (j+1) d^{r-2-j} \sum_{i=0}^{j} b^{j-i} c^i C_j^i A_{\overline{m}+j+1-i-k,\overline{n}+1+i-l} \\ & + \sum_{i=0}^{r} b^{r-i} c^i C_r^i A_{\overline{m}+r-i,\overline{n}+i} \\ & + rq \sum_{i=0}^{r-1} b^{r-1-i} c^i C_{r-1}^i A_{\overline{m}+r-1-i-k,\overline{n}+i-l}. \end{aligned} \tag{2.103}$$

Proof. In view of (2.65), for $i \in \{\overline{m}-k, \overline{m}-k+1, \dots, \overline{m}+r\}$ and $j \in \{\overline{n}-l, \overline{n}-l+1, \dots, \overline{n}+r\}$, we have

$$\begin{aligned} dA_{i,j} &= aA_{i+1,j+1} + bA_{i+1,j} + cA_{i,j+1} + p_{i,j}A_{i-k,j-l} \\ &\geq aA_{i+1,j+1} + bA_{i+1,j} + cA_{i,j+1} + qA_{i-k,j-l}. \end{aligned} \tag{2.104}$$

Hence from (2.104), we obtain

$$\begin{aligned} dA_{\overline{m},\overline{n}} &= aA_{\overline{m}+1,\overline{n}+1} + bA_{\overline{m}+1,\overline{n}} + cA_{\overline{m},\overline{n}+1} + p_{\overline{m},\overline{n}}A_{\overline{m}-k,\overline{n}-l} \\ &\geq aA_{\overline{m}+1,\overline{n}+1} + bA_{\overline{m}+1,\overline{n}} + cA_{\overline{m},\overline{n}+1} + qA_{\overline{m}-k,\overline{n}-l}, \\ dA_{\overline{m}+1,\overline{n}} &= aA_{\overline{m}+2,\overline{n}+1} + bA_{\overline{m}+2,\overline{n}} + cA_{\overline{m}+1,\overline{n}+1} + p_{\overline{m}+1,\overline{n}}A_{\overline{m}+1-k,\overline{n}-l} \\ &\geq aA_{\overline{m}+2,\overline{n}+1} + bA_{\overline{m}+2,\overline{n}} + cA_{\overline{m}+1,\overline{n}+1} + qA_{\overline{m}+1-k,\overline{n}-l}, \\ dA_{\overline{m},\overline{n}+1} &= aA_{\overline{m}+1,\overline{n}+2} + bA_{\overline{m}+1,\overline{n}+1} + cA_{\overline{m},\overline{n}+2} + p_{\overline{m},\overline{n}+1}A_{\overline{m}-k,\overline{n}+1-l} \\ &\geq aA_{\overline{m}+1,\overline{n}+2} + bA_{\overline{m}+1,\overline{n}+1} + cA_{\overline{m},\overline{n}+2} + qA_{\overline{m}-k,\overline{n}+1-l}, \\ dA_{\overline{m}-k,\overline{n}-l} &= aA_{\overline{m}+1-k,\overline{n}+1-l} + bA_{\overline{m}+1-k,\overline{n}-l} + cA_{\overline{m}-k,\overline{n}+1-l} + p_{\overline{m}-k,\overline{n}-l}A_{\overline{m}-2k,\overline{n}-2l} \\ &\geq aA_{\overline{m}+1-k,\overline{n}+1-l} + bA_{\overline{m}+1-k,\overline{n}-l} + cA_{\overline{m}-k,\overline{n}+1-l} + qA_{\overline{m}-2k,\overline{n}-2l}. \end{aligned} \tag{2.105}$$

Thus, from (2.105), we obtain

$$\begin{aligned} d^2A_{\overline{m},\overline{n}} \geq{} & adA_{\overline{m}+1,\overline{n}+1} + abA_{\overline{m}+2,\overline{n}+1} + acA_{\overline{m}+1,\overline{n}+2} \\ & + b^2A_{\overline{m}+2,\overline{n}} + 2bcA_{\overline{m}+1,\overline{n}+1} + c^2A_{\overline{m},\overline{n}+2} \\ & + ap_{\overline{m},\overline{n}}A_{\overline{m}+1-k,\overline{n}+1-l} + b(p_{\overline{m}+1,\overline{n}} + p_{\overline{m},\overline{n}})A_{\overline{m}+1-k,\overline{n}-l} \\ & + c(p_{\overline{m},\overline{n}+1} + p_{\overline{m},\overline{n}})A_{\overline{m}-k,\overline{n}+1-l} + p_{\overline{m},\overline{n}}p_{\overline{m}-k,\overline{n}-l}A_{\overline{m}-2k,\overline{n}-2l}. \end{aligned} \tag{2.106}$$

Then we obtain

$$\begin{aligned} d^2A_{\overline{m},\overline{n}} \geq{} & adA_{\overline{m}+1,\overline{n}+1} + a\sum_{i=0}^{1} b^{1-i}c^iA_{\overline{m}+2-i,\overline{n}+1+i} + aqA_{\overline{m}+1-k,\overline{n}+1-l} \\ & + \sum_{j=0}^{2} b^{2-j}c^jC_2^jA_{\overline{m}+2-j,\overline{n}+j} + 2q\sum_{j=0}^{1} b^{1-j}c^jC_1^jA_{\overline{m}+1-j-k,\overline{n}+j-l}. \end{aligned} \tag{2.107}$$

In view of the following equality, for any positive integers m, n, and r,

$$\begin{aligned}
&\sum_{i=0}^{r} b^{r-i}c^iC_r^i\left(bA_{m+r+1-i,n+i}+cA_{m+r-i,n+1+i}\right)\\
&\quad= b^{r+1}A_{m+r+1,n}+\sum_{i=1}^{r} b^{r+1-i}c^iC_r^iA_{m+r+1-i,n+i}\\
&\qquad+\sum_{i=0}^{r-1} b^{r-i}c^{i+1}C_r^iA_{m+r-i,n+1+i}+c^{r+1}A_{m,n+r+1}\\
&\quad= b^{r+1}A_{m+r+1,n}+c^{r+1}A_{m,n+r+1}\\
&\qquad+\sum_{i=1}^{r} b^{r+1-i}c^i\left(C_r^i+C_r^{i-1}\right)A_{m+r+1-i,n+i}\\
&\quad= \sum_{i=0}^{r+1} b^{r+1-i}c^iC_{r+1}^iA_{m+r+1-i,n+i},
\end{aligned} \tag{2.108}$$

and (2.105), we can obtain

$$\begin{aligned}
d^3A_{\overline{m},\overline{n}} \geq{}& a\sum_{j=0}^{2} d^{2-j}\sum_{i=0}^{j} b^{j-i}c^iC_j^iA_{\overline{m}+j+1-i,\overline{n}+1+i}\\
&+aq\sum_{j=0}^{1}(j+1)d^{1-j}\sum_{i=0}^{j} b^{j-i}c^iC_j^iA_{\overline{m}+j+1-i-k,\overline{n}+1+i-l}\\
&+\sum_{i=0}^{3} b^{3-i}c^iC_3^iA_{\overline{m}+3-i,\overline{n}+i}+3q\sum_{i=0}^{2} b^{2-i}c^iC_2^iA_{\overline{m}+2-i-k,\overline{n}+i-l}.
\end{aligned} \tag{2.109}$$

By induction, (2.103) follows. The proof is completed. □

From Lemma 2.22, we can obtain the following corollaries.

Corollary 2.23. *Assume that* $k>0$ *and* $l>0$, *and, for* $\overline{m}\geq 3k$ *and* $\overline{n}\geq 3l$, $\{A_{m,n}\}$ *is a solution of* (2.65) *such that* $A_{m,n}>0$ *for* $m\in\{\overline{m}-3k,\overline{m}-3k+1,\ldots,\overline{m}+l+1\}$ *and* $n\in\{\overline{n}-3l,\overline{n}-3l+1,\ldots,\overline{n}+k+1\}$ *and* $p_{m,n}\geq q\geq 0$ *for* $m\in\{\overline{m}-2k,\overline{m}-2k+1,\ldots,\overline{m}+l\}$ *and* $n\in\{\overline{n}-2l,\overline{n}-2l+1,\ldots,\overline{n}+k\}$. *Then*

$$\begin{aligned}
&\left(adb^{k-1}c^{l-1}C_{k+l-2}^{l-1}+b^kc^lC_{k+l}^l\right)A_{\overline{m},\overline{n}}\\
&\qquad\leq\left\{d^{k+l}-qb^{k-1}c^{l-1}\left(a(k+l-1)C_{k+l-2}^{l-1}+\frac{bc(k+l)}{d}C_{k+l}^l\right)\right\}A_{\overline{m}-k,\overline{n}-l}.
\end{aligned} \tag{2.110}$$

Proof. From Lemma 2.22, we have

$$
\begin{aligned}
d^{k+l}A_{\overline{m}-k,\overline{n}-l} \geq\ & a\sum_{j=0}^{k+l-1} d^{k+l-1-j}\sum_{i=0}^{j} b^{j-i}c^{i}C_j^{i}A_{\overline{m}-k+j+1-i,\overline{n}-l+1+i} \\
&+ aq\sum_{j=0}^{k+l-2}(j+1)d^{k+l-2-j}\sum_{i=0}^{j} b^{j-i}c^{i}C_j^{i}A_{\overline{m}-k+j+1-i-k,\overline{n}-l+1+i-l} \\
&+ \sum_{i=0}^{k+l} b^{k+l-i}c^{i}C_{k+l}^{i}A_{\overline{m}-k+k+l-1+1-i,\overline{n}-l+i} \\
&+ (k+l)q\sum_{i=0}^{k+l-1} b^{k+l-1-i}c^{i}C_{k+l-1}^{i}A_{\overline{m}+l-1-i-k,\overline{n}-2l+i} \\
\geq\ & adb^{k-1}c^{l-1}C_{k+l-2}^{l-1}A_{\overline{m},\overline{n}} + aq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1}A_{\overline{m}-k,\overline{n}-l} \\
&+ b^{k}c^{l}C_{k+l}^{l}A_{\overline{m},\overline{n}} + (k+l)qb^{k}c^{l-1}C_{k+l-1}^{l-1}A_{\overline{m}-k,\overline{n}-l-1} \\
&+ (k+l)qb^{k-1}c^{l}C_{k+l-1}^{l}A_{\overline{m}-k-1,\overline{n}-l}.
\end{aligned}
\tag{2.111}
$$

From Lemma 2.21 and the above inequality, we obtain (2.110). The proof is completed. □

Corollary 2.24. *Assume that for integers* $\overline{m} \geq 2k+l$ *and* $\overline{n} \geq 2l+k$, $\{A_{m,n}\}$ *is a solution of* (2.65) *such that* $A_{m,n} > 0$ *for* $m \in \{\overline{m}-2k-1, \overline{m}-2k, \dots, \overline{m}+k+l+2\}$ *and* $n \in \{\overline{n}-2l-1, \overline{n}-2l, \dots, \overline{n}+l+k+2\}$ *and* $p_{m,n} \geq q \geq 0$ *for* $m \in \{\overline{m}-k-1, \overline{m}-k, \dots, \overline{m}+k+l\}$ *and* $n \in \{\overline{n}-l-1, \overline{n}-l, \dots, \overline{n}+l+k\}$. *If* $k > 0$ *and* $l > 0$, *then*

$$
\begin{aligned}
&\left(d^{k+l+1} - adq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1} - (k+l+1)qb^{k}c^{l}C_{k+l}^{l}\right)A_{\overline{m}-1,\overline{n}+1} \\
&\qquad \geq (k+l+1)qb^{k+1}c^{l-1}C_{k+l}^{l-1}A_{\overline{m},\overline{n}},
\end{aligned}
\tag{2.112}
$$

$$
\begin{aligned}
&\left(d^{k+l+1} - adq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1} - (k+l+1)qb^{k}c^{l}C_{k+l}^{l}\right)A_{\overline{m}+1,\overline{n}-1} \\
&\qquad \geq (k+l+1)qb^{k-1}c^{l+1}C_{k+l}^{l+1}A_{\overline{m},\overline{n}}.
\end{aligned}
\tag{2.113}
$$

Proof. From (2.65), we have

$$
dA_{\overline{m},\overline{n}} \geq qA_{\overline{m}-k,\overline{n}-l} \tag{2.114}
$$

for $m \in \{\overline{m}-k-1, \overline{m}-k, \ldots, \overline{m}+k+l\}$ and $n \in \{\overline{n}-l-1, \overline{n}-l, \ldots, \overline{n}+k+l\}$. From Lemma 2.22 and (2.114), we obtain

$$\begin{aligned} d^{k+l+1}A_{\overline{m}-1,\overline{n}+1} \geq\ & aq\sum_{j=0}^{k+l-1}(j+1)d^{k+l-1-j}\sum_{i=0}^{j}b^{j-i}c^iC_j^iA_{\overline{m}-1+j+1-i-k,\overline{n}+1+1+i-l} \\ & + (k+l+1)q\sum_{i=0}^{k+l}b^{k+l-i}c^iC_{k+l}^iA_{\overline{m}-1+k+l-i-k,\overline{n}+1+i-l} \\ \geq\ & adq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1}A_{\overline{m}-1,\overline{n}+1} \\ & + q(k+l+1)b^{k+1}c^{l-1}C_{k+l}^{l-1}A_{\overline{m},\overline{n}} + q(k+l+1)b^kc^lC_{k+l}^lA_{\overline{m}-1,\overline{n}+1}. \end{aligned} \tag{2.115}$$

Hence (2.112) holds. Similarly, (2.113) holds. The proof is completed. □

Corollary 2.25. *Assume that for integers* $\overline{m} \geq 2k+l$ *and* $\overline{n} \geq 2l+k$, $\{A_{m,n}\}$ *is a solution of* (2.65) *such that* $A_{m,n} > 0$ *for* $m \in \{\overline{m}-2k, \overline{m}-2k+1, \ldots, \overline{m}+k+l+2\}$ *and* $n \in \{\overline{n}-2l, \overline{n}-2l+1, \ldots, \overline{n}+l+k+2\}$ *and* $p_{m,n} \geq q \geq 0$ *for* $m \in \{\overline{m}-k, \overline{m}-k+1, \ldots, \overline{m}+k+l+1\}$ *and* $n \in \{\overline{n}-l, \overline{n}-l+1, \ldots, \overline{n}+l+k+1\}$. *For* $k > 0$ *and* $l > 0$, *then*

$$\begin{aligned} & \left(d^{k+l} - aq(k+l)b^{k-1}c^{l-1}C_{k+l-2}^{l-1}\right)A_{\overline{m}+1,\overline{n}+1} \\ & \quad \geq \left(qd^{-1}b^kc^lC_{k+l}^l + q(k+l)b^kc^{l-1}C_{k+l-1}^{l-1}\right)A_{\overline{m}+1,\overline{n}} \\ & \qquad + \left(qd^{-1}b^{k-1}c^{l+1}C_{k+l}^{l+1} + q(k+l)b^{k-1}c^lC_{k+l-1}^l\right)A_{\overline{m},\overline{n}+1}. \end{aligned} \tag{2.116}$$

Proof. From Lemma 2.22 and (2.114), we have

$$\begin{aligned} d^{k+l}A_{\overline{m}+1,\overline{n}+1} \geq\ & a\sum_{j=0}^{k+l-1}d^{k+l-1-j}\sum_{i=0}^{j}b^{j-i}c^iC_j^iA_{\overline{m}+1+j+1-i,\overline{n}+1+1+i} \\ & + aq\sum_{j=0}^{k+l-2}(j+1)d^{k+l-2-j}\sum_{i=0}^{j}b^{j-i}c^iC_j^iA_{\overline{m}+1+j+1-i-k,\overline{n}+1+1+i-l} \\ & + \sum_{i=0}^{k+l}b^{k+l-i}c^iC_{k+l}^iA_{\overline{m}+1+k+l-i,\overline{n}+1+i} \\ & + (k+l)q\sum_{i=0}^{k+l-1}b^{k+l-1-i}c^iC_{k+l-1}^iA_{\overline{m}+1+k+l-1-i-k,\overline{n}+1+i-l} \end{aligned}$$

$$
\begin{aligned}
&\geq aqb^{k-1}c^{l-1}C_{k+l-2}^{l-1}A_{\overline{m}+1,\overline{n}+1}\\
&\quad + aq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1}A_{\overline{m}+1,\overline{n}+1}\\
&\quad + qd^{-1}b^{k}c^{l}C_{k+l}^{l}A_{\overline{m}+1,\overline{n}} + qd^{-1}b^{k-1}c^{l+1}C_{k+l}^{l+1}A_{\overline{m},\overline{n}+1}\\
&\quad + q(k+l)b^{k}c^{l-1}C_{k+l-1}^{l-1}A_{\overline{m}+1,\overline{n}} + q(k+l)b^{k-1}c^{l}C_{k+l-1}^{l}A_{\overline{m},\overline{n}+1}.
\end{aligned}
\tag{2.117}
$$

Hence (2.116) holds. The proof is completed. □

Lemma 2.26. *Assume that the conditions of Lemma 2.22 hold and $q = 0$. Then*

$$
\begin{aligned}
d^{r}A_{\overline{m},\overline{n}} &\geq a\sum_{j=0}^{r-1} d^{r-j-1}\sum_{i=0}^{j} b^{j-i}c^{i}C_{j}^{i}A_{\overline{m}+j+1-i,\overline{n}+1+i}\\
&\quad + a\sum_{u=0}^{r-2} d^{r-u-2}\sum_{j=0}^{u} b^{u-j}c^{j}\left\{\sum_{s=0}^{u-j}\sum_{t=0}^{j} p_{\overline{m}+s,\overline{n}+t}\right\}\times A_{\overline{m}+u+1-j-k,\overline{n}+1+j-l}\\
&\quad +\sum_{i=0}^{r} b^{r-i}c^{i}C_{r}^{i}A_{\overline{m}+k_1-i,\overline{n}+i}+\sum_{j=0}^{r-1} b^{r-1-j}c^{j}\left\{\sum_{s=0}^{r-1-j}\sum_{t=0}^{j} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m}+r-1-j-k,\overline{n}+j-l}.
\end{aligned}
\tag{2.118}
$$

Proof. As in the proof of Lemma 2.22, we know that inequality (2.106) holds. Then (2.118) holds for $r = 2$. By induction, we obtain (2.118). The proof is completed. □

Corollary 2.27. *Assume that $k > 0$ and $l > 0$, and for $\overline{m} \geq 3k + l$ and $\overline{n} \geq 3l + k$, $\{A_{m,n}\}$ is a solution of (2.65) such that $A_{m,n} > 0$ for $m \in \{\overline{m}-2k-1, \overline{m}-2k, \ldots, \overline{m}+k+l+2\}$ and $n \in \{\overline{n}-2l-1, \overline{n}-2l, \ldots, \overline{n}+k+l+2\}$, $p_{m,n} \geq 0$ for $m \in \{\overline{m}-k-1, \overline{m}-k, \ldots, \overline{m}+k+l+1\}$ and $n \in \{\overline{n}-l-1, \overline{n}-l, \ldots, \overline{n}+k+l+1\}$, and $\sum_{i=m-k}^{m-1}\sum_{j=n-l}^{n-1} p_{i,j} \geq \overline{q} \geq 0$ for $m \in \{\overline{m}, \overline{m}+1, \ldots, \overline{m}+k+1\}$ and $n \in \{\overline{n}, \overline{n}+1, \ldots, \overline{n}+l+1\}$. Then*

$$
\{d^{k+l+1} - \overline{q}b^{k-1}c^{l-1}(ad+bc)\}A_{\overline{m}-1,\overline{n}+1} \geq \overline{q}b^{k+1}c^{l-1}A_{\overline{m},\overline{n}}, \tag{2.119}
$$

$$
\{d^{k+l+1} - \overline{q}b^{k-1}c^{l-1}(ad+bc)\}A_{\overline{m}+1,\overline{n}-1} \geq \overline{q}b^{k-1}c^{l+1}A_{\overline{m},\overline{n}}. \tag{2.120}
$$

Proof. From Lemma 2.26, we have

$$\begin{aligned}
d^{k+l+1}A_{\overline{m}-1,\overline{n}+1} &\geq a\sum_{u=0}^{k+l-1} d^{k+l-1-u}\sum_{j=0}^{u} b^{u-j}c^{j}\left\{\sum_{s=0}^{u-j}\sum_{t=0}^{j} p_{\overline{m}+s,\overline{n}+t}\right\} \\
&\quad \times A_{\overline{m}-1+u+1-j-k,\overline{n}+1+1+j-l} \\
&\quad + \sum_{j=0}^{k+l} b^{k+l-j}c^{j}\left\{\sum_{s=0}^{k+l-j}\sum_{t=0}^{j} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m}-1+k+l-j-k,\overline{n}+1+j-l} \\
&\geq adb^{k-1}c^{l-1}\left\{\sum_{s=0}^{k-1}\sum_{t=0}^{l-1} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m}-1,\overline{n}+1} \\
&\quad + b^{k+1}c^{l-1}\left\{\sum_{s=0}^{k-1}\sum_{t=0}^{l-1} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m},\overline{n}} \\
&\quad + b^{k}c^{l}\left\{\sum_{s=0}^{k}\sum_{t=0}^{l} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m}-1,\overline{n}+1} \\
&\geq \overline{q}b^{k-1}c^{l-1}(ad+bc)A_{\overline{m}-1,\overline{n}+1} + \overline{q}b^{k+1}c^{l-1}A_{\overline{m},\overline{n}}.
\end{aligned} \tag{2.121}$$

Hence (2.119) holds. Similarly, (2.120) holds. The proof is completed. □

Corollary 2.28. *Assume that the conditions of Corollary 2.27 hold. Then*

$$(d^{k+l} - a\overline{q}b^{k-1}c^{l-1})A_{\overline{m}+1,\overline{n}+1} \geq \overline{q}b^{k}c^{l-1}A_{\overline{m}+1,\overline{n}} + \overline{q}b^{k-1}c^{l}A_{\overline{m},\overline{n}+1}. \tag{2.122}$$

Proof. From Lemma 2.26, we have

$$\begin{aligned}
d^{k+l}A_{\overline{m}+1,\overline{n}+1} &\geq a\sum_{u=0}^{k+l-2} d^{k+l-2-u}\sum_{j=0}^{u} b^{u-j}c^{j}\left\{\sum_{s=0}^{u-j}\sum_{t=0}^{j} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m}+1+u+1-j-k,\overline{n}+1+1+j-l} \\
&\quad + \sum_{j=0}^{k+l-1} b^{k+l-1-j}c^{j}\left\{\sum_{s=0}^{k+l-1-j}\sum_{t=0}^{j} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m}+1+k+l-1-j-k,\overline{n}+1+j-l} \\
&\geq ab^{k-1}c^{l-1}\left\{\sum_{s=0}^{k-1}\sum_{t=0}^{l-1} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m}+1,\overline{n}+1} + b^{k}c^{l-1}\left\{\sum_{s=0}^{k}\sum_{t=0}^{l-1} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m}+1,\overline{n}} \\
&\quad + b^{k-1}c^{l}\left\{\sum_{s=0}^{k-1}\sum_{t=0}^{l} p_{\overline{m}+s,\overline{n}+t}\right\}A_{\overline{m},\overline{n}+1} \\
&\geq a\overline{q}b^{k-1}c^{l-1}A_{\overline{m}+1,\overline{n}+1} + \overline{q}b^{k}c^{l-1}A_{\overline{m}+1,\overline{n}} + \overline{q}b^{k-1}c^{l}A_{\overline{m},\overline{n}+1}.
\end{aligned} \tag{2.123}$$

Hence (2.122) holds. The proof is completed. □

Define the set E of real numbers as follows:

$$E = \{\lambda > 0 \mid d - \lambda p_{m,n} > 0 \text{ eventually}\}. \tag{2.124}$$

Lemma 2.29. *Assume that $p_{m,n} \geq 0$ eventually and there exists a constant $M > 0$ such that*

$$\sup_{\lambda \in E,\, m \geq S,\, n \geq T} \lambda \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \lambda p_{i,j}) \right)^{\xi} < M \tag{2.125}$$

for all sufficiently large positive integers S and T, where ξ is a positive constant. Then the set E defined in (2.124) *is bounded.*

Proof. The lemma holds obviously if $\limsup_{m,n\to\infty} p_{m,n} > 0$. If

$$\limsup_{m,n\to\infty} p_{m,n} = 0 \tag{2.126}$$

and the set E is unbounded, then there exist $\lambda_0 \in E$ and $\lambda_0 > C = 2M/d^{\xi kl}$ such that for any sufficiently large positive integers S and T,

$$\sup_{m \geq S,\, n \geq T} \lambda_0 \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \lambda_0 p_{i,j}) \right)^{\xi} < M. \tag{2.127}$$

Since $\limsup_{m,n\to\infty} p_{m,n} = 0$, then there exist S and T such that

$$\left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \lambda_0 p_{i,j}) \right)^{\xi} \geq \frac{d^{\xi kl}}{2} \quad \forall m \geq S,\ n \geq T. \tag{2.128}$$

Hence

$$\sup_{m \geq S,\, n \geq T} \lambda_0 \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \lambda_0 p_{i,j}) \right)^{\xi} > \frac{2Md^{\xi kl}}{2d^{\xi kl}} = M, \tag{2.129}$$

which contradicts (2.127). Thus the set E is bounded. The proof is completed. □

For every eventually positive solution $A = \{A_{m,n}\}$ of (2.65), we define the set S of the positive reals as follows:

$$S(A) = \{\lambda > 0 \mid aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - (d - \lambda p_{m,n})A_{m,n} \leq 0 \text{ eventually}\}. \tag{2.130}$$

In view of (2.65) and Lemma 2.21, we have

$$\begin{aligned} &aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + p_{m,n}\frac{b^k c^l}{d^{k+l}}A_{m,n} \\ &\quad \le aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} - dA_{m,n} + p_{m,n}A_{m-k,n-l} = 0, \end{aligned} \tag{2.131}$$

which implies that

$$0 < aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \le \left(d - \frac{b^k c^l}{d^{k+l}}p_{m,n}\right)A_{m,n} \text{ eventually.} \tag{2.132}$$

Hence $b^k c^l / d^{k+l} \in S(A)$, that is, $S(A)$ is nonempty. It is easy to see that for any $\lambda \in S(A)$, we have eventually

$$aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \le (d - \lambda p_{m,n})A_{m,n}, \tag{2.133}$$

that is, $(d - \lambda p_{m,n})A_{m,n} > 0$ eventually, and then $\lambda \in E$, which leads to $S(A) \subset E$.

Theorem 2.30. *Assume that there exists a positive constant $\overline{q} > 0$ such that*
(i)

$$\sum_{i=m-k}^{m-1}\sum_{j=n-l}^{n-1} p_{i,j} \ge \overline{q} \text{ eventually;} \tag{2.134}$$

(ii) *there exist $S, T \in N_1$ such that*

$$\sup_{\lambda \in E,\, m \ge S,\, n \ge T} \lambda\left(\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}(d - \lambda p_{i,j})\right)^{1/k} < \beta^l\left(\frac{b}{d}\right)^k \quad \text{for } 0 < k \le l \tag{2.135}$$

or

$$\sup_{\lambda \in E,\, m \ge S,\, n \ge T} \lambda\left(\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}(d - \lambda p_{i,j})\right)^{1/l} < \alpha^k\left(\frac{c}{d}\right)^l \quad \text{for } k \ge l > 0, \tag{2.136}$$

where

$$\begin{aligned} \alpha &= \frac{bd^{k+l}}{d^{k+l} - a\overline{q}b^{k-1}c^{l-1}} + \frac{\overline{q}d^{k+l}b^{k+1}c^l}{\left(d^{k+l} - a\overline{q}b^{k-1}c^{l-1}\right)\left(d^{k+l+1} - \overline{q}b^{k-1}c^{l-1}(ad+bc)\right)}, \\ \beta &= \frac{cd^{k+l}}{d^{k+l} - a\overline{q}b^{k-1}c^{l-1}} + \frac{\overline{q}d^{k+l}b^k c^{l+1}}{\left(d^{k+l} - a\overline{q}b^{k-1}c^{l-1}\right)\left(d^{k+l+1} - \overline{q}b^{k-1}c^{l-1}(ad+bc)\right)}. \end{aligned} \tag{2.137}$$

Then every solution of (2.65) *oscillates.*

Proof. Suppose to the contrary, let $\{A_{m,n}\}$ be an eventually positive solution of (2.65). From Lemma 2.29, the sets E and $S(A)$ are bounded. Let $\mu \in S(A)$. By Corollaries 2.27-2.28, we have

$$(d - \mu p_{m,n})A_{m,n} \geq aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \geq \alpha A_{m+1,n}, \tag{2.138}$$

$$(d - \mu p_{m,n})A_{m,n} \geq aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \geq \beta A_{m,n+1}. \tag{2.139}$$

From (2.138), for all large m and n,

$$\begin{gathered} \alpha A_{m,n} \leq (d - \mu p_{m-1,n})A_{m-1,n}, \\ \alpha A_{m-1,n} \leq (d - \mu p_{m-2,n})A_{m-2,n}, \\ \vdots \\ \alpha A_{m-\sigma+1,n} \leq (d - \mu p_{m-\sigma,n})A_{m-\sigma,n}. \end{gathered} \tag{2.140}$$

Hence, we have

$$\alpha^k A_{m,n} \leq \prod_{i=m-k}^{m-1} (d - \mu p_{i,n})A_{m-k,n}. \tag{2.141}$$

Similarly, from (2.139), we obtain

$$\beta^l A_{m-k,n} \leq \prod_{j=n-l}^{n-1} (d - \mu p_{m-k,j})A_{m-k,n-l}. \tag{2.142}$$

From (2.141), we have

$$\alpha^k A_{m,n-j} \leq \left(\prod_{i=m-k}^{m-1} (d - \mu p_{i,n-j}) \right) A_{m-k,n-j}, \quad j = 0, 1, \ldots, l. \tag{2.143}$$

Hence by Lemma 2.21, we obtain

$$\begin{aligned} \left(\frac{c}{d}\right)^{l^2} (\alpha^k A_{m,n})^l &\leq \left(\frac{c}{d}\right)^{l(l-1)/2} \alpha^{kl} A_{m,n-1}A_{m,n-2} \cdots A_{m,n-l} \\ &\leq \prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \mu p_{i,j})A^l_{m-k,n-l}. \end{aligned} \tag{2.144}$$

Similarly, from (2.142), we obtain

$$\left(\frac{b}{d}\right)^{k^2} (\beta^l A_{m,n})^k \leq \prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \mu p_{i,j})A^k_{m-k,n-l}. \tag{2.145}$$

Therefore, we have

$$\left(\frac{b}{d}\right)^k \beta^l A_{m,n} \leq \left(\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}(d-\mu p_{i,j})\right)^{1/k} A_{m-k,n-l} \quad \text{for } k \leq l, \tag{2.146}$$

$$\left(\frac{c}{d}\right)^l \alpha^k A_{m,n} \leq \left(\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}(d-\mu p_{i,j})\right)^{1/l} A_{m-k,n-l} \quad \text{for } k > l. \tag{2.147}$$

If $k \leq l$, then in view of (2.146) and (2.65), we obtain

$$\beta^l \left(\frac{b}{d}\right)^k \sup_{m\geq M,\, n\geq N}\left(\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}(d-\mu p_{i,j})\right)^{-1/k} \in S(A). \tag{2.148}$$

On the other hand, (2.135) implies that there exists $\theta \in (0,1)$ such that

$$\sup_{\lambda\in E,\, m\geq M,\, n\geq N} \lambda\left(\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}(d-\lambda p_{i,j})\right)^{1/k} \leq \theta\beta^l\left(\frac{b}{d}\right)^k < \beta^l\left(\frac{b}{d}\right)^k. \tag{2.149}$$

Hence, we have

$$\sup_{m\geq M,\, n\geq N} \mu\left(\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}(d-\mu p_{i,j})\right)^{1/k} \leq \theta\beta^l\left(\frac{b}{d}\right)^k. \tag{2.150}$$

In view of (2.148) and (2.150), we obtain $\mu/\theta \in S(A)$. Repeating the above procedure, we obtain

$$\mu\left(\frac{1}{\theta}\right)^r \in S(A), \quad r = 1,2,\ldots, \tag{2.151}$$

which contradicts the boundedness of $S(A)$.

The second result can be proved similarly. The proof is completed. □

Corollary 2.31. *Assume that for all large m and n,*

$$\frac{1}{kl}\sum_{i=m-k}^{m-1}\sum_{j=n-l}^{n-1} p_{i,j} \geq \hat{q} \geq 0 \tag{2.152}$$

and for $k \leq l$,

$$\hat{q}\beta^l\left(\frac{b}{d}\right)^k > \frac{d^{l+1}l^l}{(l+1)^{l+1}} \tag{2.153}$$

or for $k > l$,

$$\hat{q}\alpha^k \left(\frac{c}{d}\right)^l > \frac{d^{k+1}k^k}{(k+1)^{k+1}}, \tag{2.154}$$

where α and β are defined in Theorem 2.30. Then every solution of (2.65) *oscillates.*

Proof. If $k \le l$, then we see that

$$kld - \lambda \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} p_{i,j} = \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} (d - \lambda p_{i,j}) \ge kl \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \lambda p_{i,j}) \right)^{1/kl}. \tag{2.155}$$

Hence that for all large m and n,

$$\left(d - \frac{\lambda}{kl} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} p_{i,j} \right)^l \ge \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \lambda p_{i,j}) \right)^{1/k}. \tag{2.156}$$

It is easy to see that $\max_{0<\lambda<d/c} \lambda(d-c\lambda)^l = d^{1+l}l^l/c(1+l)^{1+l}$ for a positive constant c. Hence, we have

$$\begin{aligned} \beta^l \left(\frac{b}{d}\right)^k &> \frac{d^{l+1}l^l}{\hat{q}(l+1)^{l+1}} \ge \max_{\lambda>0} \lambda \left(d - \frac{\lambda}{kl} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} p_{i,j} \right)^l \\ &\ge \lambda \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} (d - \lambda p_{i,j}) \right)^{1/k}. \end{aligned} \tag{2.157}$$

Taking the supremum on both sides of (2.157), we obtain (2.135). By Theorem 2.30, every solution of (2.65) oscillates.

The second result can be obtained similarly. The proof is completed. □

Theorem 2.32. *Assume that there exists a positive constant $q > 0$ such that*
(i) *$p_{i,j} \ge q$ eventually;*
(ii) *there exist $S, T \in N_1$ such that*

$$\sup_{\lambda \in E,\, m \ge S,\, n \ge T} \lambda \left(\prod_{i=m-k}^{m-1} (d - \lambda p_{i,n}) \right) \left(\prod_{j=n-l}^{n-1} (d - \lambda p_{m-k,j}) \right) < \overline{\alpha}^k \overline{\beta}^l, \tag{2.158}$$

where

$$\theta_1 = b + \frac{aqd^{-1}b^k c^l C_{k+l}^{l} + (k+l)aqb^k c^{l-1} C_{k+l-1}^{l-1}}{d^{k+l} - aqb^{k-1}c^{l-1}C_{k+l-2}^{l-1} - aq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1}},$$

$$\theta_2 = c + \frac{aqd^{-1}b^{k-1} c^{l+1} C_{k+l}^{l+1} + (k+l)aqb^{k-1} c^{l} C_{k+l-1}^{l}}{d^{k+l} - aqb^{k-1}c^{l-1}C_{k+l-2}^{l-1} - aq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1}},$$

$$\overline{\alpha} = \theta_1 + \theta_2 \cdot \frac{(k+l+1)qb^{k+1}c^{l-1}C_{k+l}^{l-1}}{d^{k+l+1} - adq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1} - (k+l+1)qb^k c^l C_{k+l}^{l}},$$

$$\overline{\beta} = \theta_2 + \theta_1 \cdot \frac{(k+l+1)qb^{k-1}c^{l+1}C_{k+l}^{l+1}}{d^{k+l+1} - adq(k+l-1)b^{k-1}c^{l-1}C_{k+l-2}^{l-1} - (k+l+1)qb^k c^l C_{k+l}^{l}}. \tag{2.159}$$

Then every solution of (2.65) *oscillates.*

Proof. Suppose to the contrary, let $\{A_{m,n}\}$ be an eventually positive solution of (2.65). Due to condition (i), the sets E and $S(A)$ are bounded. Let $\mu \in S(A)$. By Corollaries 2.24-2.25, we have

$$(d - \mu p_{m,n})A_{m,n} \geq aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \geq \overline{\alpha}A_{m+1,n}, \tag{2.160}$$

$$(d - \mu p_{m,n})A_{m,n} \geq aA_{m+1,n+1} + bA_{m+1,n} + cA_{m,n+1} \geq \overline{\beta}A_{m,n+1}. \tag{2.161}$$

Hence, from (2.160), we have

$$\overline{\alpha}^k A_{m,n} \leq \prod_{i=m-k}^{m-1} (d - \mu p_{i,n})A_{m-k,n}. \tag{2.162}$$

Similarly, from (2.161), we obtain

$$\overline{\beta}^l A_{m-k,n} \leq \prod_{j=n-l}^{n-1} (d - \mu p_{m-k,j})A_{m-k,n-l}. \tag{2.163}$$

From (2.162) and (2.163), we have

$$\overline{\alpha}^k \overline{\beta}^l A_{m,n} \leq \prod_{i=m-k}^{m-1} (d - \mu p_{i,n}) \prod_{j=n-l}^{n-1} (d - \mu p_{m-k,j})A_{m-k,n-l}. \tag{2.164}$$

Substituting (2.164) into (2.65), we obtain

$$\overline{\alpha}^k \overline{\beta}^l \left(\sup_{m \geq S,\, n \geq T} \prod_{i=m-k}^{m-1} (d - \mu p_{i,n}) \prod_{j=n-l}^{n-1} (d - \mu p_{m-k,j}) \right)^{-1} \in S(A). \tag{2.165}$$

On the other hand, (2.158) implies that there exists $\theta \in (0,1)$ such that

$$\sup_{\lambda\in E,\, m\geq S,\, n\geq T} \lambda \prod_{i=m-k}^{m-1} (d-\lambda p_{i,n}) \prod_{j=n-l}^{n-1} (d-\lambda p_{m-k,j}) \leq \theta \overline{\alpha}^k \overline{\beta}^l < \overline{\alpha}^k \overline{\beta}^l. \tag{2.166}$$

Hence

$$\sup_{m\geq S,\, n\geq T} \prod_{i=m-k}^{m-1} (d-\mu p_{i,n}) \prod_{j=n-l}^{n-1} (d-\mu p_{m-k,j}) \leq \frac{\theta \overline{\alpha}^k \overline{\beta}^l}{\mu}. \tag{2.167}$$

In view of (2.165) and (2.167), we obtain $(\mu/\theta) \in S(A)$. Repeating the above procedure, we obtain

$$\mu\left(\frac{1}{\theta}\right)^r \in S(A), \quad r = 1, 2, \ldots, \tag{2.168}$$

which contradicts the boundedness of $S(A)$. The proof is complete. □

Corollary 2.33. *Assume that for all large m and n, $p_{m,n} \geq q > 0$ and*

$$q\overline{\alpha}^k \overline{\beta}^l > \frac{d^{k+l+1}(k+l)^{k+l}}{(k+l+1)^{k+l+1}}. \tag{2.169}$$

Then every solution of (2.65) *oscillates.*

Proof. In view of the inequality

$$\sup_{\lambda\in(0,d/q)} \lambda(d-\lambda q)^{k+l} = \frac{d^{k+l+1}(k+l)^{k+l}}{q(k+l+1)^{k+l+1}}, \tag{2.170}$$

we can see that

$$\begin{aligned}
&\sup_{\lambda\in E,\, m\geq S,\, n\geq T} \lambda \left(\prod_{i=m-k}^{m-1} (d-\lambda p_{i,n}) \right) \left(\prod_{j=n-l}^{n-1} (d-\lambda p_{m-k,j}) \right) \\
&\quad \leq \sup_{\lambda\in E} \lambda(d-\lambda q)^{k+l} \leq \sup_{\lambda\in(0,d/q)} \lambda(d-\lambda q)^{k+l} \\
&\quad = \frac{d^{k+l+1}(k+l)^{k+l}}{q(k+l+1)^{k+l+1}} < \overline{\alpha}^k \overline{\beta}^l.
\end{aligned} \tag{2.171}$$

By Theorem 2.32, every solution of (2.65) oscillates. The proof is complete. □

Theorem 2.34. *Assume that for all large m and n,* $p_{m,n} \geq 0$ *and*

$$\limsup_{m,n\to\infty} p_{m,n} > \frac{d^{k+l+1}}{b^k c^l C_{k+l}^l + adb^{k-1}c^{l-1}C_{k+l-2}^{l-1}}. \tag{2.172}$$

Then every solution of (2.65) *oscillates.*

In fact, the conclusion of Theorem 2.34 is straightforward from (2.65) and Corollary 2.23.

Remark 2.35. To compare results here with results in Section 2.5.1, we consider the equation

$$A_{m+1,n+1} + A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-k,n-l} = 0. \tag{2.173}$$

By Corollary 2.18, if

$$p_{m,n} \geq q > \frac{(k+l)^{k+l}}{(k+l+1)^{k+l+1}}, \tag{2.174}$$

then every solution of (2.173) oscillates. By Corollary 2.33, if

$$p_{m,n} \geq q > \frac{(k+l)^{k+l}}{\bar{\alpha}^k \bar{\beta}^l (k+l+1)^{k+l+1}}, \tag{2.175}$$

then every solution of (2.173) oscillates.

By the definitions of $\bar{\alpha}$ and $\bar{\beta}$ in Theorem 2.30, it is easy to see that

$$\bar{\alpha} > 1, \qquad \bar{\beta} > 1. \tag{2.176}$$

Thus, condition (2.175) improves condition (2.174) in Section 2.5.1.

2.5.3. Oscillation of PDEs with continuous arguments

In this section, we consider the partial difference equation with continuous variables

$$\begin{aligned} &p_1A(x+a,y+b) + p_2A(x+a,y) + p_3A(x,y+b) - p_4A(x,y) \\ &\quad + P(x,y)A(x-\tau,y-\sigma) = 0. \end{aligned} \tag{2.177}$$

Throughout this section we will assume that

(i) $p_i \in R$, $p_1 \geq 0$, $p_2, p_3 \geq p_4 > 0$, $P \in C(R^+ \times R^+, R^+ - \{0\})$;
(ii) $a, b, \tau, \sigma \in R$ and $a\tau > 0$, $b\sigma > 0$;

(iii) $\tau = ka + \theta$, $\sigma = lb + \eta$, where k, l are nonnegative integers, $\theta \in [0, a)$ for $a > 0$, and $\theta \in (a, 0]$ for $a < 0$, $\eta \in [0, b)$ for $b > 0$, and $\eta \in (b, 0]$ for $b < 0$;

(iv)

$$Q(x,y) = \begin{cases} \min\{P(u,v) \mid x \le u \le x+a,\ y \le v \le y+b\}, & a > 0,\ b > 0, \\ \min\{P(u,v) \mid x+a \le u \le x,\ y \le v \le y+b\}, & a < 0,\ b > 0, \\ \min\{P(u,v) \mid x \le u \le x+a,\ y+b \le v \le y\}, & a > 0,\ b < 0, \\ \min\{P(u,v) \mid x+a \le u \le x,\ y+b \le v \le y\}, & a < 0,\ b < 0, \end{cases}$$
$$\limsup_{x,y\to\infty} Q(x,y) > 0. \tag{2.178}$$

Define the set E by

$$E = \{\lambda > 0 \mid p_4 - \lambda Q(x,y) > 0 \text{ eventually}\}. \tag{2.179}$$

Lemma 2.36. *Assume that* (2.177) *has an eventually positive solution. Then the difference inequality*

$$\begin{aligned} &p_1 w(x+a, y+b) + p_2 w(x+a, y) + p_3 w(x, y+b) - p_4 w(x,y) \\ &\quad + Q(x,y) w(x - ka, y - lb) \le 0 \end{aligned} \tag{2.180}$$

has an eventually positive solution.

Proof. Let $A(x,y)$ be an eventually positive solution of (2.177). From (2.177), we have eventually

$$\begin{aligned} &p_4\big(A(x+a,y) + A(x,y+b) - A(x,y)\big) \\ &\quad < p_1 A(x+a,y+b) + p_2 A(x+a,y) + p_3 A(x,y+b) - p_4 A(x,y) < 0. \end{aligned} \tag{2.181}$$

We consider the following four cases.

Case 1. $a > 0, b > 0$.

Let

$$w(x,y) = \int_x^{x+a} \int_y^{y+b} A(u,v)\,du\,dv. \tag{2.182}$$

Then

$$\begin{aligned}\frac{\partial w(x,y)}{\partial x} &= \int_y^{y+b} \big(A(x+a,v) - A(x,v)\big)\,dv < 0,\\ \frac{\partial w(x,y)}{\partial y} &= \int_x^{x+a} \big(A(u,y+b) - A(u,y)\big)\,du < 0.\end{aligned} \tag{2.183}$$

Integrating (2.177), we have

$$\begin{aligned}p_1 &\int_x^{x+a}\int_y^{y+b} A(u+a,v+b)\,du\,dv + p_2\int_x^{x+a}\int_y^{y+b} A(u+a,v)\,du\,dv\\ &+ p_3\int_x^{x+a}\int_y^{y+b} A(u,v+b)\,du\,dv - p_4\int_x^{x+a}\int_y^{y+b} A(u,v)\,du\,dv\\ &+ \int_x^{x+a}\int_y^{y+b} P(u,v)A(u-\tau,v-\sigma)\,du\,dv = 0.\end{aligned} \tag{2.184}$$

By (2.178), (2.182), and the above equality, we obtain

$$\begin{aligned}p_1 w(x+a,y+b) &+ p_2 w(x+a,y) + p_3 w(x,y+b) - p_4 w(x,y)\\ &+ Q(x,y)w(x-\tau,y-\sigma) \le 0.\end{aligned} \tag{2.185}$$

Since $\partial w/\partial x < 0$ and $\partial w/\partial y < 0$, we have

$$w(x-\tau,y-\sigma) = w\big(x-(ka+\theta),y-(lb+\eta)\big) \ge w(x-ka,y-lb). \tag{2.186}$$

Therefore,

$$\begin{aligned}p_1 w(x+a,y+b) &+ p_2 w(x+a,y) + p_3 w(x,y+b)\\ &- p_4 w(x,y) + Q(x,y)w(x-ka,y-lb) \le 0.\end{aligned} \tag{2.187}$$

Case 2. $a < 0$, $b > 0$.

Let

$$w(x,y) = \int_{x+a}^{x}\int_y^{y+b} A(u,v)\,du\,dv. \tag{2.188}$$

Then $\partial w/\partial x > 0$ and $\partial w/\partial y < 0$. Integrating (2.177), by (2.178) and (2.188) we have

$$\begin{aligned} &p_1 w(x+a, y+b) + p_2 w(x+a, y) + p_3 w(x, y+b) - p_4 w(x, y) \\ &\quad + Q(x, y) w(x-\tau, y-\sigma) \le 0. \end{aligned} \tag{2.189}$$

Since $\partial w/\partial x > 0$ and $\partial w/\partial y < 0$, we have

$$\begin{aligned} w(x-\tau, y-\sigma) &= w\big(x-(ka+\theta), y-(lb+\eta)\big) \\ &\ge w\big(x-ka, y-(lb+\eta)\big) \ge w(x-ka, y-lb). \end{aligned} \tag{2.190}$$

Therefore,

$$\begin{aligned} &p_1 w(x+a, y+b) + p_2 w(x+a, y) + p_3 w(x, y+b) - p_4 w(x, y) \\ &\quad + Q(x, y) w(x-ka, y-lb) \le 0. \end{aligned} \tag{2.191}$$

Case 3. $a > 0$, $b < 0$.

Let

$$w(x, y) = \int_x^{x+a} \int_{y+b}^{y} A(u, v)\,du\,dv. \tag{2.192}$$

Then $\partial w/\partial x < 0$ and $\partial w/\partial y > 0$. Similarly, we can prove that the conclusion of Lemma 2.36 holds.

Case 4. $a < 0$, $b < 0$.

Let

$$w(x, y) = \int_{x+a}^{x} \int_{y+b}^{y} A(u, v)\,du\,dv. \tag{2.193}$$

Then $\partial w/\partial x > 0$ and $\partial w/\partial y > 0$. Similarly, we can prove that the conclusion of Lemma 2.36 holds. □

Theorem 2.37. *Assume that there exist* $x_1 \ge x_0$, $y_1 \ge y_0$ *either if* $k > l > 0$ *and*

$$\begin{aligned} &\sup_{\lambda \in E,\, x \ge x_1,\, y \ge y_1} \Bigg[\lambda \prod_{i=1}^{l} \big(p_4 - \lambda Q(x-ia, y-ib)\big) \\ &\qquad \times \prod_{j=1}^{k-l} \big(p_4 - \lambda Q(x-la-ja, y-lb)\big) \Bigg] < \left(p_1 + \frac{2p_2 p_3}{p_4}\right)^l p_2^{k-l}, \end{aligned} \tag{2.194}$$

or if $l > k > 0$ *and*

$$\sup_{\lambda\in E,\, x\geq x_1,\, y\geq y_1}\left[\lambda\prod_{i=1}^{k}\left(p_4-\lambda Q(x-ia,y-ib)\right)\right.$$
$$\left.\times\prod_{j=1}^{l-k}\left(p_4-\lambda Q(x-ka,y-kb-jb)\right)\right]<\left(p_1+\frac{2p_2p_3}{p_4}\right)^k p_3^{l-k}. \tag{2.195}$$

Then every solution of (2.177) *oscillates.*

Proof. Suppose to the contrary, let $A(x,y)$ be an eventually positive solution. Let $w(x,y)$ be defined as in Lemma 2.36. We define the subset $S(w)$ of the positive numbers as follows:

$$\begin{aligned} S(w)=\{\lambda>0 \mid\ & p_1w(x+a,y+b)+p_2w(x+a,y)+p_3w(x,y+b)\\ &-\left(p_4-\lambda Q(x,y)\right)w(x,y)\leq 0 \text{ eventually}\}. \end{aligned} \tag{2.196}$$

From (2.180) we have

$$\begin{aligned} &p_1w(x+a,y+b)+p_2w(x+a,y)+p_3w(x,y+b)\\ &\quad-\left(p_4-Q(x,y)\right)w(x,y)\leq 0, \end{aligned} \tag{2.197}$$

which implies $1\in S(w)$. Hence, $S(w)$ is nonempty. For $\lambda\in S(w)$, we have eventually

$$p_4-\lambda Q(x,y)>0, \tag{2.198}$$

which implies that $S(w)\subset E$. Due to the condition (i), the set E is bounded, and hence $S(w)$ is bounded. From (2.180), we have

$$p_1w(x+a,y+b)+p_2w(x+a,y)+p_3w(x,y+b)<p_4w(x,y), \tag{2.199}$$

and so

$$\begin{aligned} w(x+a,y+b)&\leq\frac{p_4}{p_2}w(x,y+b),\\ w(x+a,y+b)&\leq\frac{p_4}{p_3}w(x+a,y). \end{aligned} \tag{2.200}$$

Let $\mu \in S(w)$. Then

$$\begin{aligned}\left(p_1+\frac{2p_2p_3}{p_4}\right)&w(x+a,y+b)\\&\le p_1w(x+a,y+b)+p_2w(x+a,y)+p_3w(x,y+b)\\&\le (p_4-\mu Q(x,y))w(x,y).\end{aligned}\tag{2.201}$$

By using the similar method as in the proof of Theorem 2.15, we obtain

$$\begin{aligned}&p_1w(x+a,y+b)+p_2w(x+a,y)+p_3w(x,y+b)\\&\quad-\Bigg\{p_4-Q(x,y)\left(p_1+\frac{2p_2p_3}{p_4}\right)^l p_2^{k-l}\\&\qquad\times\sup_{x\ge x_1,\,y\ge y_1}\Bigg[\Bigg(\prod_{i=1}^{l}(p_4-\mu Q(x-ia,y-ib))\\&\qquad\qquad\times\prod_{j=1}^{k-l}(p_4-\mu Q(x-la-ja,y-lb))\Bigg)^{-1}\Bigg]\Bigg\}w(x,y)\le 0\quad\text{for } k>l,\\&p_1w(x+a,y+b)+p_2w(x+a,y)+p_3w(x,y+b)\\&\quad-\Bigg\{p_4-Q(x,y)\left(p_1+\frac{2p_2p_3}{p_4}\right)^k p_3^{l-k}\\&\qquad\times\sup_{x\ge x_1,\,y\ge y_1}\Bigg[\Bigg(\prod_{i=1}^{k}(p_4-\mu Q(x-ia,y-ib))\\&\qquad\qquad\times\prod_{j=1}^{l-k}(p_4-\mu Q(x-ka,y-kb-jb))\Bigg)^{-1}\Bigg]\Bigg\}w(x,y)\le 0\quad\text{for } l>k.\end{aligned}\tag{2.202}$$

From (2.202) we obtain

$$\begin{aligned}&\left(p_1+\frac{2p_2p_3}{p_4}\right)^l p_2^{k-l}\sup_{x\ge x_1,y\ge y_1}\\&\quad\times\Bigg[\Bigg(\prod_{i=1}^{l}(p_4-\mu Q(x-ia,y-ib))\\&\qquad\times\prod_{j=1}^{k-l}(p_4-\mu Q(x-la-ja,y-lb))\Bigg)^{-1}\Bigg]\in S(w)\quad\text{for } k>l,\end{aligned}$$

$$\left(p_1 + \frac{2p_2p_3}{p_4}\right)^k p_3^{l-k} \sup_{x\geq x_1,\, y\geq y_1} \times \left[\left(\prod_{i=1}^{k}(p_4 - \mu Q(x-ia, y-ib)) \times \prod_{j=1}^{l-k}(p_4 - \mu Q(x-ka, y-kb-jb))\right)^{-1}\right] \in S(w) \quad \text{for } l > k. \tag{2.203}$$

On the other hand, (2.194) implies that there exists $\alpha_1 \in (0,1)$ such that for $k > l$, we have

$$\begin{aligned} \sup_{\lambda\in E,\, x\geq x_1,\, y\geq y_1} & \lambda \prod_{i=1}^{l}(p_4 - \lambda Q(x-ia, y-ib)) \\ & \times \prod_{j=1}^{k-l}(p_4 - \lambda Q(x-la-ja, y-lb)) \\ & \leq \alpha_1\left(p_1 + \frac{2p_2p_3}{p_4}\right)^l p_2^{k-l}, \end{aligned} \tag{2.204}$$

and (2.195) implies that there exists $\alpha_1 \in (0,1)$ such that for $l > k$, we have

$$\begin{aligned} \sup_{\lambda\in E,\, x\geq x_1,\, y\geq y_1} & \lambda \prod_{i=1}^{k}(p_4 - \lambda Q(x-ia, y-ib)) \\ & \times \prod_{j=1}^{l-k}(p_4 - \lambda Q(x-ka, y-kb-jb)) \\ & \leq \alpha_1\left(p_1 + \frac{2p_2p_3}{p_4}\right)^k p_3^{l-k}. \end{aligned} \tag{2.205}$$

In particular, (2.204) and (2.205) lead to (when $\lambda = \mu$), respectively,

$$\left(p_1 + \frac{2p_2p_3}{p_4}\right)^l p_2^{k-l} \sup_{x\geq x_1,\, y\geq y_1} \times \left[\left(\prod_{i=1}^{l}(p_4 - \mu Q(x-ia, y-ib)) \times \prod_{j=1}^{k-l}(p_4 - \mu Q(x-la-ja, y-lb))\right)^{-1}\right] \geq \frac{\mu}{\alpha_1} \quad \text{for } k > l, \tag{2.206}$$

$$\left(p_1 + \frac{2p_2p_3}{p_4}\right)^k p_3^{l-k} \sup_{x\geq x_1,\, y\geq y_1} \times\left[\left(\prod_{i=1}^{k}\left(p_4 - \mu Q(x-ia, y-ib)\right) \times \prod_{j=1}^{l-k}\left(p_4 - \mu Q(x-ka, y-kb-jb)\right)\right)^{-1}\right] \geq \frac{\mu}{\alpha_1} \quad \text{for } l > k. \tag{2.207}$$

Since $\mu^* \in S(w)$ and $\mu' \leq \mu^*$ imply that $\mu' \in S(w)$, it follows from (2.204) and (2.206) for $k > l$, (2.205) and (2.207) for $l > k$ that $\mu/\alpha_1 \in S(w)$. Repeating the above argument with μ replaced by μ/α_1, we get $\mu/\alpha_1\alpha_2 \in S(w)$ where $\alpha_2 \in (0,1)$. Continuing in this way, we obtain

$$\frac{\mu}{\prod_{i=1}^{\infty}\alpha_i} \in S(w), \tag{2.208}$$

where $\alpha_i \in (0,1)$. This contradicts the boundedness of S. The proof is complete. □

Corollary 2.38. *Assume that either for $k > l > 0$,*

$$\liminf_{x,y\to\infty} Q(x,y) = q > p_4^{k+1}\left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l} p_2^{l-k}\frac{k^k}{(k+1)^{k+1}}, \tag{2.209}$$

or for $l > k > 0$,

$$\liminf_{x,y\to\infty} Q(x,y) = q > p_4^{l+1}\left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-k} p_3^{k-l}\frac{l^l}{(l+1)^{l+1}}. \tag{2.210}$$

Then every solution of (2.177) *oscillates.*

Proof. We see that

$$\max_{p_4/q>\lambda>0} \lambda(p_4 - \lambda q)^k = \frac{p_4^{k+1}k^k}{q(k+1)^{k+1}}. \tag{2.211}$$

Hence (2.209) and (2.210) imply that (2.194) and (2.195) hold. By Theorem 2.37, every solution of (2.177) oscillates. The proof is complete. □

Theorem 2.39. *Assume that there exist* $x_1 \geq x_0$, $y_1 \geq y_0$ *either if* $k > l > 0$,

$$\sup_{\lambda\in E, x\geq x_1, y\geq y_1} \lambda\left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}\left(p_4 - \lambda Q(x - ia - ja, y - ib)\right)\right]^{1/(k-l)} < \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{l}\left(\frac{p_2}{p_4}\right)^{(1/2)(k-l+1)}, \tag{2.212}$$

or if $l > k > 0$,

$$\sup_{\lambda\in E, x\geq x_1, y\geq y_1} \lambda\left[\prod_{j=1}^{l-k}\prod_{i=1}^{k}\left(p_4 - \lambda Q(x - ia, y - ib - jb)\right)\right]^{1/(l-k)} < \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{k}\left(\frac{p_3}{p_4}\right)^{(1/2)(l-k+1)}. \tag{2.213}$$

Then every solution of (2.177) *oscillates.*

Proof. If $k > l$, we have

$$w(x, y) \leq \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l}\prod_{i=1}^{l}\left(p_4 - \mu Q(x - ia, y - ib)\right)w(x - la, y - lb). \tag{2.214}$$

By (2.200) and (2.214), we have

$$\begin{aligned} w(x - ja, y) &\leq \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l} \\ &\quad\times\prod_{i=1}^{l}\left(p_4 - \mu Q(x - ia - ja, y - ib)\right)w(x - la - ja, y - lb) \\ &\leq \left[\left(p_1+\frac{2p_2p_3}{p_4}\right)^{-l}\prod_{i=1}^{l}\left(p_4-\mu Q(x-ia-ja, y-ib)\right)\right]w(x-ka, y-lb) \end{aligned} \tag{2.215}$$

for $j = 1, 2, \ldots, k - l$. In view of

$$w^{k-l}(x, y) \leq \prod_{j=1}^{k-l}\left(\frac{p_4}{p_2}\right)^{j} w(x - ja, y) \tag{2.216}$$

and (2.215), we obtain

$$w^{k-l}(x,y) = \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l(k-l)} \left(\frac{p_4}{p_2}\right)^{(1/2)(k-l+1)(k-l)} \times \left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}\left(p_4 - \mu Q(x-ia-ja, y-ib)\right)\right] w^{k-l}(x-ka, y-lb). \tag{2.217}$$

That is,

$$w(x,y) \leq \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l} \left(\frac{p_4}{p_2}\right)^{(1/2)(k-l+1)} \times \left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}\left(p_4 - \mu Q(x-ia-ja, y-ib)\right)\right]^{1/(k-l)} w(x-ka, y-lb). \tag{2.218}$$

Similarly, if $l > k$, we have

$$w(x,y) \leq \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-k} \left(\frac{p_4}{p_3}\right)^{(1/2)(l-k+1)} \times \left[\prod_{j=1}^{l-k}\prod_{i=1}^{k}\left(p_4 - \mu Q(x-ia, y-ib-jb)\right)\right]^{1/(l-k)} w(x-ka, y-lb). \tag{2.219}$$

The rest of the proof is similar to that of Theorem 2.37, and thus, is omitted. □

Corollary 2.40. *Assume that either for $k > l > 0$,*

$$\liminf_{x,y\to\infty} \frac{1}{(k-l)l}\sum_{j=1}^{k-l}\sum_{i=1}^{l} Q(x-ia-ja, y-ib) > \frac{p_4^{l+1}l^l}{(l+1)^{l+1}}\left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l}\left(\frac{p_4}{p_2}\right)^{(1/2)(k-l+1)}, \tag{2.220}$$

or for $l > k > 0$,

$$\liminf_{x,y\to\infty} \frac{1}{(l-k)k} \sum_{j=1}^{l-k} \sum_{i=1}^{k} Q(x-ia, y-ib-jb) > \frac{p_4^{k+1}k^k}{(k+1)^{k+1}} \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-k} \left(\frac{p_4}{p_3}\right)^{(1/2)(l-k+1)}. \tag{2.221}$$

Then every solution of (2.177) *oscillates.*

Proof. Since

$$\max_{p_4/c>\lambda>0} \lambda(p_4-\lambda c)^l = \frac{p_4^{l+1}l^l}{c(l+1)^{l+1}} \tag{2.222}$$

let

$$c = \frac{1}{(k-l)l} \sum_{j=1}^{k-l} \sum_{i=1}^{l} Q(x-ia-ja, y-ib). \tag{2.223}$$

Then

$$\begin{aligned}
&\lambda\left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}\left(p_4-\lambda Q(x-ia-ja,y-ib)\right)\right]^{1/(k-l)} \\
&\quad\le \frac{\lambda}{(k-l)l}\left[\sum_{j=1}^{k-l}\sum_{i=1}^{l}\left(p_4-\lambda Q(x-ia-ja,y-ib)\right)\right]^{l} \\
&\quad\le \lambda\left[p_4-\frac{\lambda}{(k-l)l}\sum_{j=1}^{k-l}\sum_{i=1}^{l}Q(x-ia-ja,y-ib)\right]^{l} \\
&\quad\le p_4^{l+1}\frac{l^l}{(l+1)^{l+1}}\left[\frac{1}{(k-l)l}\sum_{j=1}^{k-l}\sum_{i=1}^{l}Q(x-ia-ja,y-ib)\right]^{-1} \\
&\quad\le \left(p_1+\frac{2p_2p_3}{p_4}\right)^{l}\left(\frac{p_2}{p_4}\right)^{(1/2)(k-l+1)}.
\end{aligned} \tag{2.224}$$

Similarly, we have

$$\lambda\left[\prod_{j=1}^{l-k}\prod_{i=1}^{k}\left(p_4-\lambda Q(x-ia,y-ib-jb)\right)\right]^{1/(l-k)} \le \left(p_1+\frac{2p_2p_3}{p_4}\right)^{k}\left(\frac{p_3}{p_4}\right)^{(1/2)(l-k+1)}. \tag{2.225}$$

By Theorem 2.39, every solution of (2.177) oscillates. The proof is complete. □

Theorem 2.41. *Assume that there exist* $x_1 \geq x_0$, $y_1 \geq y_0$ *such that if* $k = l > 0$,

$$\sup_{\lambda \in E, x \geq x_1, y \geq y_1} \lambda \prod_{i=1}^{k} (p_4 - \lambda Q(x - ia, y - ib)) < \left(p_1 + \frac{2p_2p_3}{p_4}\right)^k. \tag{2.226}$$

Then every solution of (2.177) *oscillates.*

Proof. Let $\mu \in S(w)$. Then from (2.200), we have

$$w(x, y) \leq \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-k} \prod_{i=1}^{k} (p_4 - \mu Q(x - ia, y - ib)) w(x - ka, y - kb). \tag{2.227}$$

The rest of the proof is similar to that of Theorem 2.39, and thus, is omitted. □

Since

$$\max_{p_4/q > \lambda > 0} \lambda (p_4 - \lambda q)^k = \frac{p_4^{k+1} k^k}{q(k + 1)^{k+1}}, \tag{2.228}$$

we have the following result.

Corollary 2.42. *Assume that* $k = l > 0$ *and that*

$$\liminf_{x,y \to \infty} Q(x, y) = q > \frac{p_4^{k+1} k^k}{(k + 1)^{k+1}} \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-k}. \tag{2.229}$$

Then every solution of (2.177) *oscillates.*

Theorem 2.43. *Assume that there exist* $x_1 \geq x_0$, $y_1 \geq y_0$ *such that if* $k, l > 0$ *and*

$$\sup_{\lambda \in E, x \geq x_1, y \geq x_1} \lambda \prod_{i=1}^{k} (p_4 - \lambda Q(x - ia, y)) \prod_{j=1}^{l} (p_4 - \lambda Q(x - ka, y - jb)) < p_2^k p_3^l, \tag{2.230}$$

then every solution of (2.177) *oscillates.*

Proof. Let $\mu \in S(w)$. Then

$$
\begin{aligned}
w(x,y) &\le \frac{1}{p_2}\left(p_4-\mu Q(x-a,y)\right)w(x-a,y)\\
&\le \left(\frac{1}{p_2}\right)^k \prod_{i=1}^{k}\left(p_4-\mu Q(x-ia,y)\right)w(x-ka,y)\\
&\le \left(\frac{1}{p_2}\right)^k \left(\frac{1}{p_3}\right)\prod_{i=1}^{k}\left(p_4-\mu Q(x-ia,y)\right)\left(p_4-\mu Q(x-ia,y-b)\right)\\
&\quad\times w(x-ka,y-b)\\
&\le \left(\frac{1}{p_2}\right)^k \left(\frac{1}{p_3}\right)^l\prod_{i=1}^{k}\left(p_4-\mu Q(x-ia,y)\right)\prod_{j=1}^{l}\left(p_4-\mu Q(x-ia,y-jb)\right)\\
&\quad\times w(x-ka,y-lb).
\end{aligned}
\tag{2.231}
$$

The rest of the proof is similar to that of Theorem 2.15, and thus, is omitted. □

Since

$$
\max_{p_4/q>\lambda>0}\lambda(p_4-\lambda q)^{k+l} = \frac{p_4^{k+l+1}(k+l)^{k+l}}{q(k+l+1)^{k+l+1}},
\tag{2.232}
$$

and (2.230), we have the following result.

Corollary 2.44. *Assume that $k,l>0$ and that*

$$
\liminf_{x,y\to\infty} Q(x,y) = q > \frac{p_4^{k+l+1}(k+l)^{k+l}}{p_2^k p_3^l (k+l+1)^{k+l+1}}.
\tag{2.233}
$$

Then every solution of (2.177) *oscillates.*

Theorem 2.45. *Assume that there exist $x_1 \ge x_0$, $y_1 \ge y_0$ such that if $k,l>0$ and*

$$
\sup_{\lambda\in E,\, x\ge x_1,\, y\ge y_1} \lambda\left[\prod_{j=1}^{l}\prod_{i=1}^{k}\left(p_4-\lambda Q(x-ia,y-jb)\right)\right]^{1/l} < p_2^k\left(\frac{p_3}{p_4}\right)^{(1/2)(l+1)},
\tag{2.234}
$$

or

$$
\sup_{\lambda\in E,\, x\ge x_1,\, y\ge y_1} \lambda\left[\prod_{i=1}^{k}\prod_{j=1}^{l}\left(p_4-\lambda Q(x-ia,y-jb)\right)\right]^{1/k} < p_3^l\left(\frac{p_2}{p_4}\right)^{(1/2)(k+1)},
\tag{2.235}
$$

then every solution of (2.177) *oscillates.*

Proof. Let $\mu \in S(w)$. Then, we have eventually

$$p_2 w(x+a,y) \le \big(p_4 - \mu Q(x,y)\big)w(x,y), \tag{2.236}$$

$$p_3 w(x,y+b) \le \big(p_4 - \mu Q(x,y)\big)w(x,y). \tag{2.237}$$

By (2.236), we get

$$\begin{aligned} w(x,y) &\le \frac{1}{p_2}\big(p_4 - \mu Q(x-a,y)\big)w(x-a,y) \\ &\le \cdots \le \left(\frac{1}{p_2}\right)^k \prod_{i=1}^{k}\big(p_4 - \mu Q(x-ia,y)\big)w(x-ka,y). \end{aligned} \tag{2.238}$$

Hence

$$\begin{aligned} w(x,y-jb) &\le \frac{1}{p_2^k}\prod_{i=1}^{k}\big(p_4 - \mu Q(x-ia,y-jb)\big)w(x-ka,y-jb) \\ &\le \left[\frac{1}{p_2^k}\prod_{i=1}^{k}\big(p_4-\mu Q(x-ia,y-jb)\big)\right]w(x-ka,y-lb), \quad j=1,2,\dots,l, \end{aligned} \tag{2.239}$$

and so

$$\begin{aligned} w^l(x,y) &\le \prod_{j=1}^{l}\left(\frac{p_4}{p_3}\right)^j w(x,y-jb) \\ &\le \prod_{j=1}^{l}\left\{\left(\frac{p_4}{p_3}\right)^j\left[\frac{1}{p_2^k}\prod_{i=1}^{k}\big(p_4 - \mu Q(x-ia,y-jb)\big)\right]w(x-ka,y-lb)\right\} \\ &= \frac{1}{p_2^{kl}}\left(\frac{p_4}{p_3}\right)^{(1/2)l(l+1)}\left[\prod_{j=1}^{l}\prod_{i=1}^{k}\big(p_4-\mu Q(x-ia,y-jb)\big)\right]w^l(x-ka,y-lb), \end{aligned} \tag{2.240}$$

that is,

$$w(x,y) \le \left[\frac{1}{p_2^{kl}}\left(\frac{p_4}{p_3}\right)^{l(l+1)/2}\prod_{j=1}^{l}\prod_{i=1}^{k}\big(p_4 - \mu Q(x-ia,y-jb)\big)\right]^{1/l} w(x-ka,y-lb). \tag{2.241}$$

Similarly, we have

$$w(x,y) \le \left(\frac{1}{p_3}\right)^l \prod_{j=1}^{l} \left(p_4 - \mu Q(x, y - jb)\right) w(x, y - lb),$$

$$\begin{aligned} w^k(x,y) &\le \prod_{i=1}^{k} \left(\frac{p_4}{p_2}\right)^i w(x - ia, y) \\ &\le \frac{1}{p_3^{lk}} \left(\frac{p_4}{p_2}\right)^{(1/2)k(k+1)} \left[\prod_{i=1}^{k}\prod_{j=1}^{l} \left(p_4 - \mu Q(x - ia, y - jb)\right)\right] w^k(x - ka, y - lb), \end{aligned} \tag{2.242}$$

that is,

$$w(x,y) \le \left[\frac{1}{p_3^{lk}} \left(\frac{p_4}{p_2}\right)^{k(k+l)/2} \prod_{i=1}^{k}\prod_{j=1}^{l} \left(p_4 - \mu Q(x - ia, y - jb)\right)\right]^{1/k} w(x - ka, y - lb). \tag{2.243}$$

The rest of the proof is similar to that of Theorem 2.37, and thus, is omitted. □

Corollary 2.46. *Assume that*

$$\liminf_{x,y\to\infty} \frac{1}{kl} \sum_{j=1}^{l}\sum_{i=1}^{k} Q(x - ia, y - ib) > p_2^{-k} \left(\frac{p_4}{p_3}\right)^{(l+1)/2} \frac{k^k}{(k+1)^{k+1}}, \tag{2.244}$$

or

$$\liminf_{x,y\to\infty} \frac{1}{lk} \sum_{i=1}^{k}\sum_{j=1}^{l} Q(x - ia, y - ib) > p_3^{-l} \left(\frac{p_4}{p_2}\right)^{(k+1)/2} \frac{l^l}{(l+1)^{l+1}}. \tag{2.245}$$

Then every solution of (2.177) *oscillates.*

Example 2.47. Consider the partial difference equation with continuous variables of the form

$$\begin{gathered} A\left(x + \frac{1}{2}, y - 1\right) + \frac{1}{e} A\left(x + \frac{1}{2}, y\right) + e^2 A(x, y - 1) \\ - A(x,y) + (e+1) A(x - 1, y + 2) = 0. \end{gathered} \tag{2.246}$$

It is easy to see that (2.246) satisfies the conditions of Corollary 2.42, so every solution of this equation is oscillatory. In fact, $A(x,y) = (-e)^{2x+y}$ is such a solution.

2.6. Linear PDEs with several delays

2.6.1. Equations with nonnegative coefficients

Consider the partial difference equation with several delays

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} = 0, \quad m,n \in N_0, \tag{2.247}$$

where $\{p_{m,n}^{(i)}\}$ is a double real sequence with $p_{m,n}^{(i)} \geq 0$ for all large $m, n, k_i, l_i \in N_1$, $i = 1, 2, \ldots, u$, and

$$p_{m,n}^{(i)} \geq p_i \in [0, \infty), \quad \liminf_{m,n\to\infty} p_{m,n}^{(i)} = p_i, \quad i = 1, 2, \ldots, u. \tag{2.248}$$

Then the corresponding limiting equation of (2.247) is

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_i A_{m-k_i,n-l_i} = 0, \quad m,n \in N_0. \tag{2.249}$$

The characteristic equation of (2.249) is

$$\lambda + \mu - 1 + \sum_{i=1}^{u} p_i \lambda^{-k_i} \mu^{-l_i} = 0. \tag{2.250}$$

First we define a sequence $\{\lambda_l\}_{l=1}^{\infty}$ by

$$\lambda_1 = 1, \qquad \lambda_{l+1} = 1 - \sum_{i=1}^{u} p_i \lambda_l^{-k_i-l_i}, \quad l = 1, 2, \ldots, \tag{2.251}$$

where $p_i \geq 0$, $i = 1, 2, \ldots, u$.

The following lemma will be used to prove our main results.

Lemma 2.48. *Assume that the sequence* $\{\lambda_l\}$ *is defined by* (2.251). *Then* $\lambda_* \leq \lambda_l \leq 1$ *and* $\lim_{l\to\infty} \lambda_l = \lambda_*$, *where* λ_* *is the largest root of the equation*

$$\lambda = 1 - \sum_{i=1}^{u} p_i \lambda^{-k_i-l_i} \tag{2.252}$$

on $(0, 1]$.

The proof is simple and thus omitted.

In the following, we consider the linear partial difference inequalities

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \le 0, \quad m,n \in N_0, \tag{2.253}$$

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \ge 0, \quad m,n \in N_0. \tag{2.254}$$

Assume that p_i, $i = 1,2,\ldots,u$ are sufficiently small such that the equation

$$2\lambda - 1 + \sum_{i=1}^{u} p_i \lambda^{-k_i-l_i} = 0 \tag{2.255}$$

has positive roots on $(0,1/2)$. Hence (2.250) has positive roots, which implies that (2.249) has nonoscillatory solutions. We will show sufficient conditions for the oscillation of (2.247) in this case.

Theorem 2.49. *Assume that* (2.248) *holds. Further, assume that*

$$\limsup_{m,n\to\infty} \sum_{i=1}^{u} \left(\lambda_*^{-k_i-l_i} p_{m,n}^{(i)} + \lambda_*^{1-k_i-l_i} \left(p_{m+1,n}^{(i)} + p_{m,n+1}^{(i)}\right)\right) > 1, \tag{2.256}$$

where λ_* *is the largest root of* (2.252) *on* $(0,1]$. *Then*

(i) *equation* (2.253) *has no eventually positive solutions;*
(ii) *equation* (2.254) *has no eventually negative solutions;*
(iii) *every solution of* (2.247) *oscillates.*

Proof. It is sufficient to prove that (i), (ii), and (iii) follow from (i). Assume, for the sake of contradiction, that $\{A_{m,n}\}$ is an eventually positive solution of (2.253). Then, there exist m_1 and n_1 such that $A_{m,n} > 0$ and $A_{m-k_i,n-l_i} > 0$, $i = 1,2,\ldots,u$ for $m \ge m_1$, $n \ge n_1$. Therefore, from (2.253), we have

$$A_{m+1,n} < A_{m,n}, \quad A_{m,n+1} < A_{m,n}, \quad m \ge m_1,\ n \ge n_1, \tag{2.257}$$

which gives

$$A_{m-k_i,n} \ge \lambda_1^{-k_i} A_{m,n} \quad \text{for } m \ge m_1 + k_i,\ n \ge n_1. \tag{2.258}$$

Hence, we have

$$A_{m-k_i,n-l_i} \ge \lambda_1^{-k_i-l_i} A_{m,n} \quad \text{for } m \ge m_1 + k_i,\ n \ge n_1 + l_i. \tag{2.259}$$

Using now (2.259) and (2.247), we have

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_i \lambda_1^{-k_i-l_i} A_{m,n} \le 0. \tag{2.260}$$

Hence, we have

$$A_{m+1,n} \le A_{m,n}\left(1-\sum_{i=1}^{u} p_i\lambda_1^{-k_i-l_i}\right) = \lambda_2 A_{m,n} \quad \text{for } m \ge m_1+k_i,\ n \ge n_1+l_i,$$
$$A_{m,n+1} \le A_{m,n}\left(1-\sum_{i=1}^{u} p_i\lambda_1^{-k_i-l_i}\right) = \lambda_2 A_{m,n} \quad \text{for } m \ge m_1+k_i,\ n \ge n_1+l_i. \tag{2.261}$$

Hence

$$A_{m-k_i,n-l_i} \ge \lambda_2^{-k_i-l_i} A_{m,n} \quad \text{for } m \ge m_1+2k_i,\ n \ge n_1+2l_i. \tag{2.262}$$

Repeating the above procedure, we get

$$A_{m+1,n} \le A_{m,n}\left(1-\sum_{i=1}^{u} p_i\lambda_{l-1}^{-k_i-l_i}\right) = \lambda_l A_{m,n} \tag{2.263}$$

for $m \ge m_1+(l-1)k_i$, $n \ge n_1+(l-1)l_i$, and

$$A_{m,n+1} \le A_{m,n}\left(1-\sum_{i=1}^{u} p_i\lambda_{l-1}^{-k_i-l_i}\right) = \lambda_l A_{m,n} \tag{2.264}$$

for $m \ge m_1+(l-1)k_i$, $n \ge n_1+(l-1)l_i$. Hence,

$$A_{m-k_i,n-l_i} \ge \lambda_l^{-k_i-l_i} A_{m,n} \quad \text{for } m \ge m_1+lk_i,\ n \ge n_1+ll_i, \tag{2.265}$$

where

$$\lambda_l = 1-\sum_{i=1}^{u} p_i\lambda_{l-1}^{-k_i-l_i}. \tag{2.266}$$

Since $\lim_{l\to\infty}\lambda_l = \lambda_*$, for a sequence $\{\varepsilon_l\}$ with $\varepsilon_l > 0$, and $\varepsilon_l \to 0$ as $l \to \infty$, by (2.263), (2.264), and (2.265) there exists a double sequence $\{m_l, n_l\}$ such that $m_l, n_l \to \infty$ as $l \to \infty$ and

$$A_{m+1,n} \le (\lambda_*+\varepsilon_l)A_{m,n} \quad \text{for } m \ge m_l,\ n \ge n_l, \tag{2.267}$$

$$A_{m,n+1} \le (\lambda_*+\varepsilon_l)A_{m,n} \quad \text{for } m \ge m_l,\ n \ge n_l, \tag{2.268}$$

$$A_{m-k_i,n-l_i} \ge (\lambda_*+\varepsilon_l)^{-k_i-l_i} A_{m,n} \quad \text{for } m \ge m_l+k_i,\ n \ge n_l+l_i. \tag{2.269}$$

From (2.247) and (2.269), we have

$$A_{m,n} \geq \sum_{i=1}^{u} p_{m,n}^{(i)} (\lambda_* + \varepsilon_l)^{1-k_i-l_i} A_{m-1,n},$$
$$A_{m,n} \geq \sum_{i=1}^{u} p_{m,n}^{(i)} (\lambda_* + \varepsilon_l)^{1-k_i-l_i} A_{m,n-1}. \tag{2.270}$$

Dividing (2.247) by $A_{m,n}$, we have

$$1 = \frac{A_{m+1,n} + A_{m,n+1}}{A_{m,n}} + \sum_{i=1}^{u} p_{m,n}^{(i)} \frac{A_{m-k_i,n-l_i}}{A_{m,n}}. \tag{2.271}$$

From (2.269)–(2.271), we have

$$1 \geq \sum_{i=1}^{u} \left((\lambda_* + \varepsilon_l)^{-k_i-l_i} p_{m,n}^{(i)} + (\lambda_* + \varepsilon_l)^{1-k_i-l_i} \left(p_{m+1,n}^{(i)} + p_{m,n+1}^{(i)}\right)\right). \tag{2.272}$$

Letting $l \to \infty$, the above inequality implies

$$\limsup_{m,n\to\infty} \sum_{i=1}^{u} \left(\lambda_*^{-k_i-l_i} p_{m,n}^{(i)} + \lambda_*^{1-k_i-l_i} \left(p_{m+1,n}^{(i)} + p_{m,n+1}^{(i)}\right)\right) \leq 1, \tag{2.273}$$

which contradicts (2.256) and completes the proof. □

Theorem 2.50. *Assume that* (2.248) *holds and* (2.256) *does not hold. If*

$$\limsup_{m,n\to\infty} \left(\sum_{i=1}^{u} \lambda_*^{-k_i-l_i} p_{m,n}^{(i)} + \frac{\sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} p_{m+1,n}^{(i)}}{1 - \sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} \left(p_{m+2,n}^{(i)} + p_{m+1,n+1}^{(i)}\right)} + \frac{\sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} p_{m,n+1}^{(i)}}{1 - \sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} \left(p_{m+1,n+1}^{(i)} + p_{m,n+2}^{(i)}\right)} \right) > 1, \tag{2.274}$$

then the conclusions of Theorem 2.49 remain.

Proof. In fact, from (2.270), we have

$$A_{m+1,n} \geq \sum_{i=1}^{u} p_{m+1,n}^{(i)} (\lambda_* + \varepsilon_l)^{1-k_i-l_i} A_{m,n} \quad \text{for } m \geq m_l,\ n \geq n_l,$$
$$A_{m,n+1} \geq \sum_{i=1}^{u} p_{m,n+1}^{(i)} (\lambda_* + \varepsilon_l)^{1-k_i-l_i} A_{m,n} \quad \text{for } m \geq m_l,\ n \geq n_l. \tag{2.275}$$

Hence

$$\begin{aligned} A_{m,n} &= A_{m+1,n} + A_{m,n+1} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \\ &\geq \sum_{i=1}^{u} \left(p_{m+1,n}^{(i)} + p_{m,n+1}^{(i)}\right)\left(\lambda_* + \varepsilon_l\right)^{1-k_i-l_i} A_{m,n} \\ &\quad + \sum_{i=1}^{u} p_{m,n}^{(i)} \left(\lambda_* + \varepsilon_l\right)^{1-k_i-l_i} A_{m-1,n} \end{aligned} \tag{2.276}$$

and hence

$$A_{m,n} \geq \frac{\sum_{i=1}^{u} \left(\lambda_* + \varepsilon_l\right)^{1-k_i-l_i} p_{m,n}^{(i)}}{1 - \sum_{i=1}^{u} \left(\lambda_* + \varepsilon_l\right)^{1-k_i-l_i} \left(p_{m+1,n}^{(i)} + p_{m,n+1}^{(i)}\right)} A_{m-1,n}. \tag{2.277}$$

Similarly,

$$A_{m,n} \geq \frac{\sum_{i=1}^{u} \left(\lambda_* + \varepsilon_l\right)^{1-k_i-l_i} p_{m,n}^{(i)}}{1 - \sum_{i=1}^{u} \left(\lambda_* + \varepsilon_l\right)^{1-k_i-l_i} \left(p_{m+1,n}^{(i)} + p_{m,n+1}^{(i)}\right)} A_{m,n-1}. \tag{2.278}$$

Substituting the above inequalities into (2.271) and letting $l \to \infty$, we obtain a contradiction with (2.274). The proof is complete. □

Since

$$\sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} p_{m+1,n}^{(i)} \geq \lambda_* \sum_{i=1}^{u} \lambda_*^{-k_i-l_i} p_i = \lambda_* \left(1 - \lambda_*\right), \tag{2.279}$$

from (2.274), we can obtain a simpler condition.

Corollary 2.51. *If* (2.274) *is replaced by*

$$\limsup_{m,n\to\infty} \sum_{i=1}^{u} \lambda_*^{-k_i-l_i} p_{m,n}^{(i)} > 2 - \frac{1}{\lambda_*^2 + \left(1 - \lambda_*\right)^2}, \tag{2.280}$$

then the conclusions of Theorem 2.49 remain.

In fact, (2.280) implies (2.274).

Theorem 2.52. *Assume that* (2.248) *holds. Further, assume that*

$$\limsup_{m,n\to\infty}\left(\frac{1}{(1-\lambda_*)^2}\sum_{i=1}^{u} p_{m,n}^{(i)}\lambda_*^{-k_i-l_i}(1-\lambda_*^{l_i+1})(1-\lambda_*^{k_i+1})+Q(m,n,\lambda_*)\right)>1, \tag{2.281}$$

where λ_* *is the largest root of* (2.252) *on* $(0,1]$ *and*

$$\begin{aligned} Q(m,n,\lambda_*) &= \frac{\sum_{i=1}^{u}\lambda_*^{1-k_i-l_i}p_{m+1,n+1}^{(i)}}{1-\sum_{i=1}^{u}\lambda_*^{1-k_i-l_i}\left(p_{m+2,n+1}^{(i)}+p_{m+1,n+2}^{(i)}\right)} \\ &\quad\times\frac{\sum_{i=1}^{u}\lambda_*^{1-k_i-l_i}p_{m+1,n}^{(i)}}{1-\sum_{i=1}^{u}\lambda_*^{1-k_i-l_i}\left(p_{m+2,n}^{(i)}+p_{m+1,n+1}^{(i)}\right)}. \end{aligned} \tag{2.282}$$

Then

(i) *equation* (2.253) *has no eventually positive solutions;*
(ii) *equation* (2.254) *has no eventually negative solutions;*
(iii) *every solution of* (2.247) *oscillates.*

Proof. It is sufficient to prove that (i), (ii), and (iii) follow from (i). Assume, for the sake of contradiction, that $\{A_{m,n}\}$ is an eventually positive solution of (2.253). Summing (2.253) in n from $n(\geq n_1)$ to ∞, we have

$$\sum_{\nu=n}^{\infty}A_{m+1,\nu}-A_{m,n}+\sum_{i=1}^{u}\sum_{\nu=n}^{\infty}p_{m,\nu}^{(i)}A_{m-k_i,\nu-l_i}\leq 0. \tag{2.283}$$

We rewrite the above inequality in the form

$$\sum_{\nu=n+1}^{\infty}A_{m+1,\nu}+A_{m+1,n}-A_{m,n}+\sum_{i=1}^{u}\sum_{\nu=n}^{\infty}p_{m,\nu}^{(i)}A_{m-k_i,\nu-l_i}\leq 0. \tag{2.284}$$

Summing it in m from $m(\geq m_1)$ to ∞, we obtain

$$\sum_{s=m}^{\infty}\sum_{\nu=n+1}^{\infty}A_{s+1,\nu}-A_{m,n}+\sum_{i=1}^{u}\sum_{s=m}^{\infty}\sum_{\nu=n}^{\infty}p_{s,\nu}^{(i)}A_{s-k_i,\nu-l_i}\leq 0. \tag{2.285}$$

From (2.277) and (2.278), we have

$$A_{m+1,n+1}\geq Q(m,n,\lambda_*+\varepsilon_l)A_{m,n}. \tag{2.286}$$

By (2.285), we get

$$
\begin{aligned}
A_{m,n} &\geq \sum_{s=m}^{\infty}\sum_{\nu=n+1}^{\infty} A_{s+1,\nu} + \sum_{i=1}^{u}\sum_{s=m}^{\infty}\sum_{\nu=n}^{\infty} p_{s,\nu}^{(i)} A_{s-k_i,\nu-l_i} \\
&\geq A_{m+1,n+1} + \sum_{i=1}^{u}\sum_{s=m}^{m+k_i}\sum_{\nu=n}^{n+l_i} p_{s,\nu}^{(i)} A_{s-k_i,\nu-l_i} \\
&\geq A_{m+1,n+1} + \sum_{i=1}^{u}\sum_{s=0}^{k_i}\sum_{\nu=0}^{l_i} p_{s+m,\nu+n}^{(i)} A_{m+s-k_i,n+\nu-l_i} \qquad (2.287) \\
&\geq A_{m+1,n+1} + A_{m,n}\sum_{i=1}^{u}\sum_{s=0}^{k_i}\sum_{\nu=0}^{l_i} p_{s+m,\nu+n}^{(i)} (\lambda_* + \varepsilon_l)^{(s-k_i)+(\nu-l_i)} \\
&= A_{m+1,n+1} + A_{m,n}\sum_{i=1}^{u}\left(\sum_{s=m}^{m+k_i}\sum_{\nu=n}^{n+l_i} p_{m,n}^{(i)} (\lambda_* + \varepsilon_l)^{(s-m-k_i)+(\nu-n-l_i)}\right).
\end{aligned}
$$

Letting $l \to \infty$, the above two inequalities imply

$$
\limsup_{m,n\to\infty}\left(\frac{1}{(1-\lambda_*)^2}\sum_{i=1}^{u} p_{m,n}^{(i)}\lambda_*^{-k_i-l_i}(1-\lambda_*^{l_i+1})(1-\lambda_*^{k_i+1}) + Q(m,n,\lambda_*)\right) \leq 1, \tag{2.288}
$$

which contradicts (2.281) and completes the proof. □

Since

$$
Q(m,n,\lambda_*) \geq \left(\frac{\lambda_*(1-\lambda_*)}{1-2\lambda_*+2\lambda_*^2}\right)^2, \tag{2.289}
$$

we can derive a simpler condition from (2.281).

Corollary 2.53. *If* (2.281) *is replaced by*

$$
\limsup_{m,n\to\infty}\frac{1}{(1-\lambda_*)^2}\sum_{i=1}^{u} p_{m,n}^{(i)}\lambda_*^{-k_i-l_i}(1-\lambda_*^{l_i+1})(1-\lambda_*^{k_i+1}) > 1 - \left(\frac{\lambda_*(1-\lambda_*)}{1-2\lambda_*+2\lambda_*^2}\right)^2, \tag{2.290}
$$

then the conclusions of Theorem 2.52 remain.

Example 2.54. Consider the partial difference equation

$$
A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-1,n-1} = 0, \quad m \geq 0,\ n \geq 0, \tag{2.291}
$$

where

$$p_{m,n} = \begin{cases} \dfrac{1}{27}, & m = n \in N_0, \\ \dfrac{1}{5}, & \text{otherwise.} \end{cases} \tag{2.292}$$

For (2.291), (2.255) is

$$2\lambda - 1 + \frac{1}{27}\lambda^{-2} = 0, \tag{2.293}$$

which has a positive root $\lambda = 1/3$. The limiting equation of (2.291) is

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \frac{1}{27}A_{m-1,n-1} = 0, \quad m \geq 0,\ n \geq 0, \tag{2.294}$$

which has a positive solution $\{A_{m,n}\} = \{3^{-(m+n)}\}$, $m, n \in N_0$. Equation (2.252) is

$$\lambda - 1 + \frac{1}{27}\lambda^{-2} = 0, \tag{2.295}$$

which has a positive root $\lambda_* = (2/3)\cos(\phi/3)$, where $\cos\phi = 1/2$. Thus, $\phi = \pi/3$ and $\lambda_* \approx 0.63$. Since $\limsup_{m,n\to\infty} p_{m,n} = 1/5$, we have

$$\limsup_{m,n\to\infty} p_{m,n} > \lambda_*^2\left(2 - \frac{1}{\lambda_*^2 + (1-\lambda_*)^2}\right) \approx 0.05. \tag{2.296}$$

By Corollary 2.51, every solution of (2.291) is oscillatory.

2.6.2. Equations with oscillatory coefficients

Consider the linear partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} = 0, \tag{2.297}$$

where $k_i, l_i \in N_1$, $k_1 > k_2 > \cdots > k_u > 0$, $l_1 > l_2 > \cdots > l_u > 0$, $p_{m,n}^{(i)}$ are real double sequences and may change sign in m, n for $i = 1, 2, \ldots, u$.

Lemma 2.55. *Assume that there exist sufficiently large M and N such that*

$$p_{m,n}^{(1)} \geq 0, \quad p_{m,n}^{(1)} + p_{m,n}^{(2)} \geq 0, \ldots, \sum_{i=1}^{u} p_{m,n}^{(i)} \geq 0 \quad \textit{for } m \geq M,\ n \geq N. \tag{2.298}$$

Further assume that for any given positive integers M and N, there exist $M_1 \geq M$, $N_1 \geq N$ *such that*

$$p_{m,n}^{(i)} \geq 0 \quad \text{for } m \in [M_1, M_1 + k_1],\ n \in [N_1, N_1 + l_1],\ i = 1, 2, \dots, u. \tag{2.299}$$

Let $\{A_{m,n}\}$ *be an eventually positive solution of* (2.297). *Then* $A_{m,n}$ *is eventually nonincreasing in m, n, and*

$$\sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \geq A_{m-k_u,n-l_u} \sum_{i=1}^{u} p_{m,n}^{(i)}. \tag{2.300}$$

Proof. Let $A_{m-k_1,n-l_1} > 0$ for $m \geq M$, $n \geq N$. By condition (2.299),

$$p_{m,n}^{(i)} \geq 0, \quad i = 1, 2, \dots, u,\ m \in [M_1, M_1 + k_1],\ n \in [N_1, N_1 + l_1]. \tag{2.301}$$

Then

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} = -\sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \leq 0 \tag{2.302}$$

for $m \in [M_1, M_1 + k_1]$, $n \in [N_1, N_1 + l_1]$.

We will show that $A_{m,n}$ is nonincreasing for $m \in [M_1 + k_1, M_1 + k_1 + k_u]$, $n \in [N_1 + l_1, N_1 + l_1 + l_u]$. In fact, $m - k_i \in [M_1, M_1 + k_1]$ for $m \in [M_1 + k_1, M_1 + k_1 + k_u]$ and $n - l_i \in [N_1, N_1 + l_1]$ for $n \in [N_1 + l_1, N_1 + l_1 + l_u]$. From (2.302), we have $A_{m-k_1,n-l_1} \geq A_{m-k_2,n-l_2} \geq \cdots \geq A_{m-k_u,n-l_u}$ for $m \in [M_1, M_1 + k_1]$, $n \in [N_1, N_1 + l_1]$. Therefore

$$\begin{aligned} A_{m+1,n} + A_{m,n+1} - A_{m,n} &= -\sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \\ &\leq -\left(p_{m,n}^{(1)} + p_{m,n}^{(2)}\right) A_{m-k_2,n-l_2} - \sum_{i=3}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \\ &\leq \cdots \leq -\left(\sum_{i=1}^{u} p_{m,n}^{(i)}\right) A_{m-k_u,n-l_u} \leq 0 \end{aligned} \tag{2.303}$$

for $m \in [M_1 + k_1, M_1 + k_1 + k_u]$, $n \in [N_1 + l_1, N_1 + l_1 + l_u]$.

Repeating the above method, it follows that $A_{m,n}$ is nonincreasing for $m \geq M_1$, $n \geq N_1$, and (2.300) holds. □

Theorem 2.56. *Suppose that the assumptions of Lemma 2.55 hold. Further, assume that*

$$\sum_{i=M}^{\infty} \sum_{j=N}^{\infty} \sum_{s=1}^{u} p_{i,j}^{(s)} = \infty. \tag{2.304}$$

Then every nonoscillatory solution of (2.297) *tends to zero as* $m, n \to \infty$.

Proof. Let $\{A_{m,n}\}$ be an eventually positive solution of (2.297). By Lemma 2.55, $A_{m,n}$ is eventually nonincreasing and hence $\lim_{m,n\to\infty} A_{m,n} = L \geq 0$ and

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + A_{m-k_u,n-l_u} \sum_{i=1}^{u} p_{m,n}^{(i)} \leq 0. \tag{2.305}$$

Summing (2.305) in n from n to ∞, we obtain

$$\sum_{i=n}^{\infty} A_{m+1,i} - A_{m,n} + \sum_{i=n}^{\infty} \sum_{s=1}^{u} p_{m,i}^{(s)} A_{m-k_s,i-l_s} \leq 0, \tag{2.306}$$

that is,

$$A_{m+1,n} - A_{m,n} + \sum_{i=n+1}^{\infty} A_{m+1,i} + \sum_{i=n}^{\infty} \sum_{s=1}^{u} p_{m,i}^{(s)} A_{m-k_s,i-l_s} \leq 0. \tag{2.307}$$

Summing (2.307) in m from m to ∞, we obtain

$$-A_{m,n} + \sum_{j=m}^{\infty} \sum_{i=n+1}^{\infty} A_{j+1,i} + \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} \sum_{s=1}^{u} p_{j,i}^{(s)} A_{j-k_s,i-l_s} \leq 0. \tag{2.308}$$

Thus

$$A_{m,n} \geq \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} \sum_{s=1}^{u} p_{j,i}^{(s)} A_{j-k_s,i-l_s}. \tag{2.309}$$

If $L > 0$, (2.309) contradicts (2.304). The proof is complete. □

Define the subset of positive reals as follows:

$$E = \left\{ \lambda > 0 \mid 1 - \lambda \sum_{i=1}^{u} p_{m,n}^{(i)} > 0 \text{ eventually} \right\}. \tag{2.310}$$

Given an eventually positive solution $\{A_{m,n}\}$ of (2.297), we define the subset $S(A)$ of the positive reals as follows:

$$S(A) = \left\{ \lambda > 0 \mid A_{m+1,n} + A_{m,n+1} - A_{m,n} \left(1 - \lambda \sum_{i=1}^{u} p_{m,n}^{(i)} \right) \leq 0 \text{ eventually} \right\}. \tag{2.311}$$

If $\lambda \in S(A)$, then $1 - \lambda \sum_{i=1}^{u} p_{m,n}^{(i)} > 0$ eventually. Therefore $S(A) \subset E$.

It is easy to see that condition

$$\limsup_{m,n\to\infty} \sum_{i=1}^{u} p_{m,n}^{(i)} > 0 \tag{2.312}$$

implies that the set E is bounded.

Theorem 2.57. *Assume that conditions of Lemma 2.55 and* (2.312) *hold. Further, assume that*

$$\sup_{\lambda\in E,\, m\geq M,\, n\geq N} \lambda\left\{\prod_{i=m-k_u}^{m-1}\prod_{j=n-l_u}^{n-1}\left(1-\lambda\sum_{s=1}^{u}p_{i,j}^{(s)}\right)\right\}^{1/\eta} < 1 \tag{2.313}$$

for some positive integers M and N, where $\eta = \min\{k_u, l_u\}$. Then every solution of (2.297) *oscillates.*

Proof. Let $\{A_{m,n}\}$ be an eventually positive solution of (2.297). Then by Lemma 2.55, $A_{m,n}$ is nonincreasing in m, n eventually and

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + A_{m-k_u,n-l_u}\sum_{i=1}^{u}p_{m,n}^{(i)} \leq 0, \tag{2.314}$$

thus we have

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u}p_{m,n}^{(i)}A_{m,n} \leq 0, \tag{2.315}$$

so

$$0 < A_{m+1,n} + A_{m,n+1} \leq \left(1-\sum_{i=1}^{u}p_{m,n}^{(i)}\right)A_{m,n}, \tag{2.316}$$

which implies that $S(A)$ is nonempty.

Let $\mu \in S(A)$, then

$$A_{m+1,n} \leq \left(1-\mu\sum_{i=1}^{u}p_{m,n}^{(i)}\right)A_{m,n} \tag{2.317}$$

and so

$$A_{m,n} \leq \prod_{r=m-k_u}^{m-1}\left(1-\mu\sum_{i=1}^{u}p_{r,n}^{(i)}\right)A_{m-k_u,n}. \tag{2.318}$$

Similarly, we have

$$A_{m,n+1} \leq \left(1-\mu\sum_{i=1}^{u}p_{m,n}^{(i)}\right)A_{m,n} \tag{2.319}$$

and so

$$A_{m,n} \leq \prod_{s=n-l_u}^{n-1}\left(1-\mu\sum_{i=1}^{u}p_{m,s}^{(i)}\right)A_{m,n-l_u}. \tag{2.320}$$

Hence

$$A_{m,n}^{l_u} \le A_{m,n-1} \cdots A_{m,n-l_u} \le \prod_{s=n-l_u}^{n-1} \prod_{r=m-k_u}^{m-1} \left(1 - \mu \sum_{i=1}^{u} p_{r,s}^{(i)}\right) A_{m-k_u,n-l_u}^{l_u}. \tag{2.321}$$

Similarly, we have

$$A_{m,n}^{k_u} \le A_{m-1,n} \cdots A_{m-k_u,n} \le \prod_{r=m-k_u}^{m-1} \prod_{s=n-l_u}^{n-1} \left(1 - \mu \sum_{i=1}^{u} p_{r,s}^{(i)}\right) A_{m-k_u,n-l_u}^{k_u}. \tag{2.322}$$

Hence, we obtain

$$A_{m,n} \le \left\{ \prod_{r=m-k_u}^{m-1} \prod_{s=n-l_u}^{n-1} \left(1 - \mu \sum_{i=1}^{u} p_{r,s}^{(i)}\right) \right\}^{1/\eta} A_{m-k_u,n-l_u}. \tag{2.323}$$

Substituting the above inequality into (2.314), we obtain

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} \left\{ 1 - \sum_{i=1}^{u} p_{m,n}^{(i)} \left[\prod_{r=m-k_u}^{m-1} \prod_{s=n-l_u}^{n-1} \left(1 - \mu \sum_{i=1}^{u} p_{r,s}^{(i)}\right) \right]^{-1/\eta} \right\} \le 0, \tag{2.324}$$

which implies that

$$\left\{ \sup_{m \ge M,\, n \ge N} \left[\prod_{r=m-k_u}^{m-1} \prod_{s=n-l_u}^{n-1} \left(1 - \mu \sum_{i=1}^{u} p_{r,s}^{(i)}\right) \right]^{1/\eta} \right\}^{-1} \in S(A). \tag{2.325}$$

From condition (2.313), there exists $\gamma \in (0,1)$ such that

$$\sup_{\lambda \in E,\, m \ge M,\, n \ge N} \lambda \left\{ \prod_{r=m-k_u}^{m-1} \prod_{s=n-l_u}^{n-1} \left(1 - \lambda \sum_{i=1}^{u} p_{r,s}^{(i)}\right) \right\}^{1/\eta} \le \gamma < 1. \tag{2.326}$$

Hence

$$\left\{ \sup_{m \ge M,\, n \ge N} \left[\prod_{r=m-k_u}^{m-1} \prod_{s=n-l_u}^{n-1} \left(1 - \mu \sum_{i=1}^{u} p_{r,s}^{(i)}\right) \right]^{1/\eta} \right\}^{-1} \ge \frac{\mu}{\gamma}, \tag{2.327}$$

so that $\mu/\gamma \in S(A)$. By induction, $\mu/\gamma^j \in S(A)$, $j = 1, 2, \ldots$. This contradicts the boundedness of $S(A)$. The proof is complete. □

Remark 2.58. The nonnegativity of all coefficients of (2.297) is not required in Theorem 2.57.

Remark 2.59. From (2.300), (2.314) implies that if every solution of

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + A_{m-k_u,n-l_u} \sum_{i=1}^{u} p_{m,n}^{(i)} = 0 \tag{2.328}$$

oscillates, then every solution of (2.297) oscillates.

From Theorem 2.57, we can derive an explicit oscillation criterion.

Corollary 2.60. *Assume that conditions of Lemma 2.55 hold. Further assume that*

$$\liminf_{m,n\to\infty} \frac{1}{k_u l_u} \sum_{i=m-k_u}^{m-1} \sum_{j=n-l_u}^{n-1} \sum_{s=1}^{u} p_{i,j}^{(s)} > \frac{\sigma^\sigma}{(1+\sigma)^{1+\sigma}}, \tag{2.329}$$

where $\sigma = \max\{k_u, l_u\}$. *Then every solution of* (2.297) *oscillates.*

Proof. Let $g(\lambda) = \lambda(1 - c\lambda)^\sigma$ for $\lambda > 0$, $c > 0$. Then

$$\max_{\lambda>0} g(\lambda) = \frac{\sigma^\sigma}{c(1+\sigma)^{1+\sigma}}. \tag{2.330}$$

Set $c = (1/k_u l_u) \sum_{r=m-k_u}^{m-1} \sum_{s=n-l_u}^{n-1} \sum_{i=1}^{u} p_{r,s}^{(i)}$. Since

$$\left\{1 - \frac{\lambda}{k_u l_u} \sum_{r=m-k_u}^{m-1} \sum_{s=n-l_u}^{n-1} p_{r,s}^{(i)}\right\}^\sigma \geq \left\{ \prod_{r=m-k_u}^{m-1} \prod_{s=n-l_u}^{n-1} \left(1 - \lambda \sum_{i=1}^{u} p_{r,s}^{(i)}\right)\right\}^{1/\eta}, \tag{2.331}$$

we obtain

$$\begin{aligned} 1 &> \frac{\sigma^\sigma}{(1+\sigma)^{1+\sigma}} \left\{ \frac{1}{k_u l_u} \sum_{r=m-k_u}^{m-1} \sum_{s=n-l_u}^{n-1} \sum_{i=1}^{u} p_{r,s}^{(i)} \right\}^{-1} \\ &\geq \lambda \left(1 - \frac{\lambda}{k_u l_u} \sum_{r=m-k_u}^{m-1} \sum_{s=n-l_u}^{n-1} \sum_{i=1}^{u} p_{r,s}^{(i)}\right)^\sigma \\ &\geq \lambda \left\{ \prod_{r=m-k_u}^{m-1} \prod_{s=n-l_u}^{n-1} \left(1 - \lambda \sum_{i=1}^{u} p_{r,s}^{(i)}\right)\right\}^{1/\eta}. \end{aligned} \tag{2.332}$$

Then the conclusion follows from Theorem 2.57. □

Example 2.61. Consider the equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \left(3 + \sin\frac{\pi}{3}m\right) A_{m-3,n-5} + \sin\frac{\pi}{3} m A_{m-1,n-2} = 0. \tag{2.333}$$

We can see that this equation satisfies Corollary 2.60, so every solution is oscillatory. In fact, $(-1)^{m+n+1}$ is an oscillatory solution.

2.6.3. Equations with positive coefficients and $p \in (0,1]$

Consider the equation

$$A_{m+1,n} + A_{m,n+1} - pA_{m,n} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} = 0, \tag{2.334}$$

where $p \in (0,1]$ and $p_{m,n}^{(i)} \geq 0$, $i = 1,2,\ldots,u$, $\sum_{i=1}^{u} p_{m,n}^{(i)} > 0$ for $m \geq N$, $n \geq N$.

Lemma 2.62. *Let $\{A_{m,n}\}$ be an eventually positive solution of* (2.334). *Then $\{A_{m,n}\}$ is eventually decreasing in m and n and for all sufficiently large m and n,*

$$A_{m+1,n} < pA_{m,n}, \qquad A_{m,n+1} < pA_{m,n}. \tag{2.335}$$

From (2.335), for positive integers k and l,

$$A_{m+k,n+l} < p^{k+l} A_{m,n}. \tag{2.336}$$

Now consider (2.334) together with difference inequalities

$$A_{m+1,n} + A_{m,n+1} - pA_{m,n} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \leq 0, \tag{2.337}$$

$$A_{m+1,n} + A_{m,n+1} - pA_{m,n} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \geq 0. \tag{2.338}$$

Theorem 2.63. *Assume that k_i and l_i are positive integers and*

(i) $\liminf_{m,n\to\infty} p_{m,n}^{(i)} = c_i > 0$, $i = 1,2,\ldots,u$;

(ii) $\sum_{i=1}^{u} p^{-1-k_i-l_i}(\limsup_{m,n\to\infty} p_{m,n}^{(i)} + 2c_i) > 1$.

Then

(a) *equation* (2.337) *has no eventually positive solutions;*

(b) *equation* (2.338) *has no eventually negative solutions;*

(c) *every solution of* (2.334) *oscillates.*

Proof. Since (2.334) is linear. To prove Theorem 2.63, it is sufficient to prove (a). Suppose to the contrary, let $\{A_{m,n}\}$ be an eventually positive solution of (2.337). Then there exist positive integers M and N such that $A_{m,n} > 0$ and $p_{m,n}^{(i)} \geq c_i - \epsilon > 0$ for $m \geq M$, $n \geq N$, where $\epsilon \in (0, \min_i c_i)$ is arbitrarily small. From (2.337), we have

$$pA_{m,n} > \sum_{i=1}^{u} (c_i - \epsilon) A_{m-k_i,n-l_i}. \tag{2.339}$$

By Lemma 2.62, we obtain

$$pA_{m,n} > A_{m-1,n-1} \sum_{i=1}^{u} p^{2-k_i-l_i}(c_i - \epsilon). \tag{2.340}$$

Hence,

$$A_{m+1,n} > A_{m,n-1} \sum_{i=1}^{u} p^{1-k_i-l_i}(c_i - \epsilon) > A_{m,n} \sum_{i=1}^{u} p^{-k_i-l_i}(c_i - \epsilon). \tag{2.341}$$

Similarly, we have

$$A_{m,n+1} > A_{m,n} \sum_{i=1}^{u} p^{-k_i-l_i}(c_i - \epsilon). \tag{2.342}$$

Hence, from (2.337), (2.341), and (2.342), we obtain

$$\begin{aligned} 0 &\geq A_{m+1,n} + A_{m,n+1} - pA_{m,n} + \sum_{i=1}^{u} p_{m,n}^{(i)} A_{m-k_i,n-l_i} \\ &> A_{m,n}\left(\sum_{i=1}^{u} p^{-k_i-l_i}\left(p_{m,n}^{(i)} + 2(c_i - \epsilon)\right) - p\right). \end{aligned} \tag{2.343}$$

Hence

$$\sum_{i=1}^{u} p^{-k_i-l_i}\left(p_{m,n}^{(i)} + 2(c_i - \epsilon)\right) < p, \tag{2.344}$$

which contradicts condition (ii). The proof is complete. □

Example 2.64. Consider the partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \left(1 + \frac{1}{n}\right)A_{m-1,n-1} + p_{m,n}A_{m-2,n-1} = 0, \tag{2.345}$$

where $p_{m,n} = (2n^3 + 7n^2 + 5n + 2)/(n(n+1)(n+2))$, $m \geq 3$, $n \geq 3$.

It is easy to see that all assumptions of Theorem 2.63 hold. Therefore every solution of (2.345) oscillates. In fact, $\{A_{m,n}\} = \{(-1)^m(1/(n+1))\}$ is such a solution.

Theorem 2.65. *Assume that (i) of Theorem 2.63 holds and*

$$\sum_{i=1}^{u} c_i \frac{(r_i + 1)^{r_i+1}}{p^{r_i+\bar{r}_i+1} r_i^{r_i}} > 1. \tag{2.346}$$

Then every solution of (2.334) *oscillates, where* $r_i = \min\{k_i, l_i\}$, $\bar{r}_i = \max\{k_i, l_i\}$, $i = 1, 2, \ldots, u$, *and* $0^0 = 1$.

Proof. Suppose to the contrary, let $\{A_{m,n}\}$ be an eventually positive solution of (2.334). Set

$$\alpha_{m,n} = \frac{A_{m,n}}{A_{m+1,n+1}} > \frac{1}{p^2}, \tag{2.347}$$

by Lemma 2.62. From (2.334), we have

$$\frac{A_{m+1,n} + A_{m,n+1}}{A_{m,n}} - p = -\sum_{i=1}^{u} p_{m,n}^{(i)} \frac{A_{m-k_i,n-l_i}}{A_{m,n}}. \tag{2.348}$$

Hence, by Lemma 2.62 and (2.348), we have

$$\begin{aligned} \frac{A_{m+1,n+1}}{A_{m,n}} - p^2 &< \frac{2A_{m+1,n+1}}{A_{m,n}} - p^2 < p\left(\frac{A_{m+1,n} + A_{m,n+1}}{A_{m,n}} - p\right) \\ &= -p\sum_{i=1}^{u} p_{m,n}^{(i)} \frac{A_{m-k_i,n-l_i}}{A_{m,n}} < -p\sum_{i=1}^{u} p_{m,n}^{(i)} p^{r_i-\bar{r}_i} \frac{A_{m-r_i,n-r_i}}{A_{m,n}}. \end{aligned} \tag{2.349}$$

We note that

$$\frac{A_{m-r_i,n-r_i}}{A_{m,n}} = \frac{A_{m-r_i,n-r_i}}{A_{m-r_{i+1},n-r_{i+1}}} \frac{A_{m-r_{i+1},n-r_{i+1}}}{A_{m-r_{i+2},n-r_{i+2}}} \cdots \frac{A_{m-1,n-1}}{A_{m,n}}. \tag{2.350}$$

Then (2.349) becomes

$$\alpha_{m,n}^{-1} - p^2 < -p\sum_{i=1}^{u} p_{m,n}^{(i)} p^{r_i-\bar{r}_i} \prod_{s=1}^{r_i} \alpha_{m-s,n-s}. \tag{2.351}$$

We claim that $\sum_{i=1}^{n} r_i^2 \neq 0$ under condition (2.346). Otherwise, from (2.346), we obtain $\sum_{i=1}^{u} c_i p^{-\bar{r}_i-1} > 1$. On the other hand, from (2.351), we have

$$-p^2 < \alpha_{m,n}^{-1} - p^2 < -\sum_{i=1}^{u} p_{m,n}^{(i)} p^{-\bar{r}_i+1} \leq -\sum_{i=1}^{u} c_i p^{-\bar{r}_i+1}. \tag{2.352}$$

Hence

$$1 > \sum_{i=1}^{u} c_i p^{-\bar{r}_i-1}. \tag{2.353}$$

This contradiction shows that $\sum_{i=1}^{n} r_i^2 \neq 0$. Since $c_i > 0$, (2.351) implies that $\alpha_{m,n}$ is bounded above. Set

$$l = \liminf_{m,n\to\infty} \alpha_{m,n}. \tag{2.354}$$

Then $l \in (1/p^2, \infty)$. From (2.351), we obtain

$$\begin{aligned} \limsup_{m,n\to\infty} (\alpha_{m,n}^{-1}) = \frac{1}{l} &< p^2 - \sum_{i=1}^{u} p^{r_i-\bar{r}_i+1} \liminf_{m,n\to\infty} \left(p_{m,n}^{(i)} \prod_{s=1}^{r_i} \alpha_{m-s,n-s} \right) \\ &\leq p^2 - \sum_{i=1}^{u} p^{r_i-\bar{r}_i+1} c_i l^{r_i}. \end{aligned} \tag{2.355}$$

Hence

$$\sum_{i=1}^{u} c_i \frac{l^{r_i+1} p^{r_i-\bar{r}_i+1}}{lp^2 - 1} \leq 1. \tag{2.356}$$

We note that

$$\liminf_{lp^2>1} \left(\frac{p^{r_i-\bar{r}_i+1} l^{r_i+1}}{lp^2 - 1} \right) = p^{-r_i-\bar{r}_i-1} \frac{(r_i+1)^{r_i+1}}{r_i^{r_i}}. \tag{2.357}$$

Combining (2.356) and (2.357), we obtain

$$\sum_{i=1}^{u} c_i \frac{(r_i+1)^{r_i+1}}{p^{r_i+\bar{r}_i+1} r_i^{r_i}} \leq 1, \tag{2.358}$$

which contradicts (2.346). The proof is complete. □

Example 2.66. Consider the partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \frac{2n^2+n-3}{n(n+1)} A_{m,n-1} + A_{m-1,n} = 0, \tag{2.359}$$

where $m \geq 1$, $n \geq 1$.

It is easy to see that (2.346) holds. By Theorem 2.65, every solution of (2.359) is oscillatory. In fact, $A_{m,n} = (-1)^m(1/n)$ is such a solution of (2.359).

2.6.4. Equations with continuous arguments

Consider the partial difference equation with continuous arguments of the form

$$\begin{aligned} &d_1 A(x+a, y+b) + d_2 A(x+a, y) + d_3 A(x, y+b) - d_4 A(x, y) \\ &\quad + \sum_{i=1}^{u} p_i(x, y) A(x - \tau_i, y - \sigma_i) = 0, \end{aligned} \tag{2.360}$$

where $p_i \in C(R^+ \times R^+, R^+)$, a, b, τ_i and σ_i, $i = 1, 2, \dots, u$ are positive, d_i, $i = 1, 2, 3$ are nonnegative and d_4 is positive.

Throughout this section, we assume that

(i) $\tau_i = k_i a + \theta_i$, $\sigma_i = l_i b + \eta_i$, where k_i, l_i are nonnegative integers, $\theta_i \in [0, a)$, $\eta_i \in [0, b)$;

(ii)

$$Q_i(x, y) = \min\{p_i(u, v) \mid x \le u \le x + a,\ y \le v \le y + b\} \tag{2.361}$$

and $\liminf_{x,y\to\infty} Q_i(x, y) = q_i \ge 0$, $i = 1, 2, \dots, u$.

The following result is obvious.

Lemma 2.67. *Let $A(x, y)$ be an eventually positive solution of* (2.360). *Set*

$$\omega(x, y) = \int_x^{x+a}\int_y^{y+b} A(u, v)\,du\,dv. \tag{2.362}$$

Then $\omega(x, y)$ is an eventually positive solution of the difference inequality

$$\begin{aligned} d_1\omega(x + a, y + b) + d_2\omega(x + a, y) + d_3\omega(x, y + b) - d_4\omega(x, y) \\ + \sum_{i=1}^{u} Q_i(x, y)\omega(x - k_i a, y - l_i b) \le 0, \end{aligned} \tag{2.363}$$

and $\partial\omega/\partial x < 0$, $\partial\omega/\partial y < 0$.

From (2.363), $d_2\omega(x + a, y) \le d_4\omega(x, y)$. Let $\lambda_1 = 0$. We have

$$\begin{aligned} \omega(x - a, y) &\ge e^{-\lambda_1}\left(\frac{d_2}{d_4}\right)\omega(x, y), \\ \omega(x - k_i a, y) &\ge e^{-k_i\lambda_1}\left(\frac{d_2}{d_4}\right)^{k_i}\omega(x, y). \end{aligned} \tag{2.364}$$

From $d_3\omega(x, y + b) \le d_4\omega(x, y)$, we have

$$\frac{d_3}{d_4}\omega(x, y + b) \le \omega(x, y). \tag{2.365}$$

Hence,

$$\begin{aligned} \omega(x, y - b) &\ge e^{-\lambda_1}\left(\frac{d_3}{d_4}\right)\omega(x, y), \\ \omega(x, y - l_i b) &\ge e^{-l_i\lambda_1}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x, y), \\ \omega(x - k_i a, y - l_i b) &\ge e^{-k_i\lambda_1}\left(\frac{d_2}{d_4}\right)^{k_i}\omega(x, y - l_i b) \\ &\ge e^{-(k_i + l_i)\lambda_1}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x, y). \end{aligned} \tag{2.366}$$

From (2.363) and (ii), we have

$$d_1\omega(x+a,y+b)+d_2\omega(x+a,y)+d_3\omega(x,y+b)-d_4\omega(x,y) + \sum_{i=1}^{u} q_i\omega(x-k_ia,y-l_ib) \le 0. \tag{2.367}$$

Hence

$$\frac{d_2}{d_4}\omega(x+a,y) \le \omega(x,y)\left(1-\frac{1}{d_4}\sum_{i=1}^{u} q_i e^{-(k_i+l_i)\lambda_1}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}\right). \tag{2.368}$$

Let

$$e^{\lambda_2} = 1-\frac{1}{d_4}\sum_{i=1}^{u} q_i e^{-(k_i+l_i)\lambda_1}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}. \tag{2.369}$$

Then

$$\frac{d_2}{d_4}\omega(x+a,y) \le e^{\lambda_2}\omega(x,y), \tag{2.370}$$

or

$$\omega(x-a,y) \ge e^{-\lambda_2}\frac{d_2}{d_4}\omega(x,y). \tag{2.371}$$

Similarly,

$$\frac{d_3}{d_4}\omega(x,y+b) \le \omega(x,y)\left(1-\frac{1}{d_4}\sum_{i=1}^{u} q_i e^{-(k_i+l_i)\lambda_1}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}\right) = e^{\lambda_2}\omega(x,y). \tag{2.372}$$

Hence

$$\omega(x,y-b) \ge e^{-\lambda_2}\frac{d_3}{d_4}\omega(x,y). \tag{2.373}$$

By induction,

$$\begin{aligned} \omega(x-a,y) &\ge e^{-\lambda_n}\frac{d_2}{d_4}\omega(x,y), \\ \omega(x,y-b) &\ge e^{-\lambda_n}\frac{d_3}{d_4}\omega(x,y), \end{aligned} \tag{2.374}$$

where

$$e^{\lambda_n} = 1-\frac{1}{d_4}\sum_{i=1}^{u} q_i e^{-(k_i+l_i)\lambda_{n-1}}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}. \tag{2.375}$$

Obviously, $\{\lambda_n\}$ is decreasing and bounded. So $\lim_{n\to\infty}\lambda_n = \lambda^*$ exists and

$$e^{\lambda^*} = 1 - \frac{1}{d_4}\sum_{i=1}^{u} q_i e^{-(k_i+l_i)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}. \tag{2.376}$$

Hence, we have

$$\omega(x-k_ia, y-l_ib) \geq e^{-(k_i+l_i)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x,y). \tag{2.377}$$

From (2.363),

$$\begin{aligned} d_4\omega(x,y) \geq{}& d_1\omega(x+a,y+b) + d_2\omega(x+a,y) + d_3\omega(x,y+b) \\ &+ \sum_{i=1}^{u} Q_i(x,y)\omega(x-k_ia,y-l_ib). \end{aligned} \tag{2.378}$$

Since

$$\omega(x-k_ia, y-l_ib) \geq e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x-a,y), \tag{2.379}$$

we have

$$\omega(x,y) \geq \frac{1}{d_4}\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x-a,y). \tag{2.380}$$

Hence

$$\omega(x+a,y) \geq \frac{1}{d_4}\sum_{i=1}^{u} Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x,y). \tag{2.381}$$

Similarly, we have

$$\omega(x,y+b) \geq \frac{1}{d_4}\sum_{i=1}^{u} Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i-1}\omega(x,y). \tag{2.382}$$

Substituting the above inequalities into (2.378), we obtain

$$\begin{aligned} d_4\omega(x,y) \geq{} & \frac{d_1}{d_4}\sum_{i=1}^{u} Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i-1}\omega(x,y) \\ & +\frac{d_2}{d_4}\sum_{i=1}^{u} Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x,y) \\ & +\frac{d_3}{d_4}\sum_{i=1}^{u} Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i-1}\omega(x,y) \\ & +\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x-a,y). \end{aligned} \tag{2.383}$$

Set

$$\begin{aligned} U(x,y) ={} & d_4-\frac{d_1}{d_4}\sum_{i=1}^{u} Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i-1} \\ & -\frac{d_2}{d_4}\sum_{i=1}^{u} Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i} \\ & -\frac{d_3}{d_4}\sum_{i=1}^{u} Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i-1}. \end{aligned} \tag{2.384}$$

Then (2.383) leads to

$$\omega(x,y) \geq \frac{1}{U(x,y)}\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i}\omega(x-a,y). \tag{2.385}$$

Similarly, we have

$$\begin{aligned} \omega(x,y) &\geq \frac{1}{U(x,y)}\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i-1}\omega(x,y-b), \\ \omega(x,y) &\geq \frac{1}{U(x,y)}\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i-2)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i-1}\omega(x-a,y-b). \end{aligned} \tag{2.386}$$

From (2.363), we have

$$\begin{aligned}
1 \geq\ & \frac{d_1}{d_4}\frac{\omega(x+a,y+b)}{\omega(x,y)} + \frac{d_2}{d_4}\frac{\omega(x+a,y)}{\omega(x,y)} + \frac{d_3}{d_4}\frac{\omega(x,y+b)}{\omega(x,y)} \\
& + \frac{1}{d_4}\sum_{i=1}^{u} Q_i(x,y)\frac{\omega(x-k_ia,y-l_ib)}{\omega(x,y)} \\
\geq\ & \frac{d_1}{d_4}\frac{1}{U(x+a,y+b)}\sum_{i=1}^{u} Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*} \\
& \times \left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i-1} \\
& + \frac{d_2}{d_4}\frac{1}{U(x+a,y)}\sum_{i=1}^{u} Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i} \\
& + \frac{d_3}{d_4}\frac{1}{U(x,y+b)}\sum_{i=1}^{u} Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i-1} \\
& + \frac{1}{d_4}\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i} \triangleq H(x,y).
\end{aligned} \tag{2.387}$$

From (2.387), we obtain the main result in this section.

Theorem 2.68. *Assume that*

$$\limsup_{x,y\to\infty} H(x,y) > 1. \tag{2.388}$$

Then every solution of (2.360) *is oscillatory.*

From (2.376), we have

$$d_4\left(1-e^{\lambda^*}\right) = \sum_{i=1}^{u} q_i e^{-(k_i+l_i)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}. \tag{2.389}$$

By (2.389), we can obtain a simpler oscillation condition from (2.388).

In view of (2.384), we have

$$
\begin{aligned}
U(x,y) \le{}& d_4 - \frac{d_1}{d_4}\sum_{i=1}^{u} q_i e^{-(k_i+l_i-2)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i-1} \\
&- \frac{d_2}{d_4}\sum_{i=1}^{u} q_i e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i-1}\left(\frac{d_3}{d_4}\right)^{l_i} \\
&- \frac{d_3}{d_4}\sum_{i=1}^{u} q_i e^{-(k_i+l_i-1)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i-1} \\
={}& d_4 - \frac{d_1}{d_4} d_4\left(1-e^{\lambda^*}\right)e^{2\lambda^*}\frac{d_4^2}{d_2d_3} \\
&- \frac{d_2}{d_4} d_4\left(1-e^{\lambda^*}\right)e^{\lambda^*}\frac{d_4}{d_2} - \frac{d_3}{d_4} d_4\left(1-e^{\lambda^*}\right)e^{\lambda^*}\frac{d_4}{d_3} \\
={}& d_4\left[1-\left(1-e^{\lambda^*}\right)\left(\frac{d_1d_4}{d_2d_3}e^{2\lambda^*}+2e^{\lambda^*}\right)\right] \\
={}& d_4\left[1-\left(1-e^{\lambda^*}\right)e^{\lambda^*}\left(\frac{d_1d_4}{d_2d_3}e^{\lambda^*}+2\right)\right].
\end{aligned}
\tag{2.390}
$$

Therefore,

$$
\begin{aligned}
H(x,y) \ge{}& \frac{1}{d_4\left[1-\left(1-e^{\lambda^*}\right)e^{\lambda^*}\left(\left(d_1d_4/d_2d_3\right)e^{\lambda^*}+2\right)\right]} \\
&\times\left\{\frac{d_1}{d_4}d_4\left(1-e^{\lambda^*}\right)e^{2\lambda^*}\frac{d_4^2}{d_2d_3}\right. \\
&\qquad\left.+\frac{d_2}{d_4}d_4\left(1-e^{\lambda^*}\right)e^{\lambda^*}\frac{d_4}{d_2}+\frac{d_3}{d_4}d_4\left(1-e^{\lambda^*}\right)e^{\lambda^*}\frac{d_4}{d_3}\right\} \\
&+\frac{1}{d_4}\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i} \\
={}& \frac{\left(1-e^{\lambda^*}\right)e^{\lambda^*}\left(\left(d_1d_4/d_2d_3\right)e^{\lambda^*}+2\right)}{1-\left(1-e^{\lambda^*}\right)e^{\lambda^*}\left(\left(d_1d_4/d_2d_3\right)e^{\lambda^*}+2\right)} \\
&+\frac{1}{d_4}\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i}.
\end{aligned}
\tag{2.391}
$$

Then we have the following simpler result.

Corollary 2.69. *Assume that*

$$\begin{aligned}\limsup_{x,y\to\infty} \frac{1}{d_4}\sum_{i=1}^{u} Q_i(x,y)e^{-(k_i+l_i)\lambda^*}\left(\frac{d_2}{d_4}\right)^{k_i}\left(\frac{d_3}{d_4}\right)^{l_i} \\ > \frac{1-2(1-e^{\lambda^*})e^{\lambda^*}((d_1d_4/d_2d_3)e^{\lambda *}+2)}{1-(1-e^{\lambda^*})e^{\lambda^*}((d_1d_4/d_2d_3)e^{\lambda^*}+2)}.\end{aligned} \tag{2.392}$$

Then every solution of (2.360) *oscillates.*

From (2.391), we have

$$H(x,y) \geq \frac{(1-e^{\lambda^*})e^{\lambda^*}((d_1d_4/d_2d_3)e^{\lambda^*}+2)}{1-(1-e^{\lambda^*})e^{\lambda^*}((d_1d_4/d_2d_3)e^{\lambda^*}+2)} + (1-e^{\lambda^*}). \tag{2.393}$$

Corollary 2.70. *If*

$$\frac{(1-e^{\lambda^*})e^{\lambda^*}((d_1d_4/d_2d_3)e^{\lambda^*}+2)}{1-(1-e^{\lambda^*})e^{\lambda^*}((d_1d_4/d_2d_3)e^{\lambda^*}+2)} + (1-e^{\lambda^*}) > 1, \tag{2.394}$$

then every solution of (2.360) *oscillates.*

Example 2.71. Consider the partial difference equation

$$\begin{aligned}A(x+2\pi,y+2\pi)+A(x+2\pi,y)+A(x,y+2\pi)-A(x,y) \\ +p(x,y)A(x-\pi,y-3\pi)=0,\end{aligned} \tag{2.395}$$

where $p(x,y) = 11/5 + \sin x + \sin y$. Then

$$Q(x,y) = \min_{\substack{x\leq u\leq x+2\pi \\ y\leq v\leq y+2\pi}} p(u,v) = \frac{1}{5}, \tag{2.396}$$

that is, $q = 1/5$. By (2.376), $e^{\lambda^*} = 1-(1/5)e^{-\lambda^*}$. Hence $e^{\lambda^*} = (1-\sqrt{1/5})/2 \approx 0.276393$, and

$$\frac{(1-e^{\lambda^*})e^{\lambda^*}(e^{\lambda^*}+2)}{1-(1-e^{\lambda^*})e^{\lambda^*}(e^{\lambda^*}+2)} + (1-e^{\lambda^*}) = 1.5594133 > 1. \tag{2.397}$$

By Corollary 2.70, every solution of (2.395) oscillates.

2.7. Frequent oscillations

In this section, we will consider the difference equation

$$aA_{m+1,n} + bA_{m,n+1} - dA_{m,n} + \sum_{i=1}^{r} p_{m,n}^{(i)} A_{m-\sigma_i,n-\tau_i} = 0, \quad m,n = 0,1,2,\ldots, \tag{2.398}$$

where a, b, and d are three positive real constants, σ_i, τ_i, and r are positive integers, and $\{p_{m,n}^{(i)}\}_{m,n=0}^{\infty}$ are real double sequences, $i = 1,2,\ldots,r$.

Since the above usual concept of oscillation does not catch all the fine details of an oscillatory sequence, a strengthened oscillation called frequent oscillation has been posed.

First, we introduce the related definitions and lemmas.

Let $Z = \{\ldots,-1,0,1,\ldots\}$, $N_k = \{k,k+1,k+2,\ldots\}$ and

$$Z^2 = \{(m,n) \mid m,n \in Z\}, \qquad N_k^2 = \{(m,n) \mid m,n \in N_k\}. \tag{2.399}$$

An element of Z^2 is called a lattice point. The union, intersection, and difference of two sets A and B of lattice points will be denoted by $A+B$ (or $A \cup B$), $A \cdot B$ (or $A \cap B$) and $A - B$ (or $A \backslash B$), respectively. Let Ω be a set of lattice points. The size of Ω is denoted by $|\Omega|$. Given integers m and n, the translation operators X^m and Y^n are defined by

$$X^m\Omega = \{(i+m,j) \in Z^2 \mid (i,j) \in \Omega\}, \qquad Y^n\Omega = \{(i,j+n) \in Z^2 \mid (i,j) \in \Omega\}, \tag{2.400}$$

respectively, and $\Omega^{(m,n)} = \{(i,j) \mid (i,j) \in \Omega,\ i \le m,\ j \le n\}$. Let α, β, and θ, δ be integers such that $\alpha \le \beta$ and $\theta \le \delta$. The union $\sum_{i=\alpha}^{\beta}\sum_{j=\theta}^{\delta} X^iY^j\Omega$ is called a derived set of Ω. Hence

$$(i,j) \in Z^2 \setminus \sum_{i=\alpha}^{\beta}\sum_{j=\theta}^{\delta} X^iY^j\Omega \Longleftrightarrow (i-k,j-l) \in Z^2 \setminus \Omega, \tag{2.401}$$

for $\alpha \le k \le \beta$ and $\theta \le l \le \delta$.

Definition 2.72. Let Ω be a set of integers. If $\limsup_{m,n\to\infty}(|\Omega^{(m,n)}|/mn)$ exists, then the limit, denoted by $\mu^*(\Omega)$, will be called the upper frequency measure of Ω. Similarly, if $\liminf_{m,n\to\infty}(|\Omega^{(m,n)}|/mn)$ exists, then the limit, denoted by $\mu_*(\Omega)$, will be called the lower frequency measure of Ω. If $\mu^*(\Omega) = \mu_*(\Omega)$, then the common limit, denoted by $\mu(\Omega)$, will be called the frequency measure of Ω.

Definition 2.73. Let $A = \{A_{m,n} \mid m \geq -u,\ n \geq -v\}$ be a real double sequence and let $\lambda \in [0,1]$ be a constant. If $\mu^*(A \leq 0) \leq \lambda$, then A is said to be frequently positive of upper degree λ, and if $\mu^*(A \geq 0) \leq \lambda$, then A is said to be frequently negative of upper degree λ. The sequence A is said to be frequently oscillatory of upper degree λ if it is neither frequently positive nor frequently negative of the same upper degree λ. The concept of frequently positive of lower degree, and so forth, is similarly defined by means of μ_*. If a sequence A is frequently oscillatory of upper degree 0, it is said to be frequently oscillatory.

Obviously, if a double sequence is eventually positive (or eventually negative), then it is frequently positive (or frequently negative). Thus, if the sequence is frequently oscillatory, then it is oscillatory.

We will adopt the usual notation for level sets of a double sequence, that is, let $A : \Omega \to R$ be a double sequence, then the set $\{(m,n) \in \Omega \mid A_{m,n} \leq c\}$ will be denoted by $(A \leq c)$ or $(A_{m,n} \leq c)$, where c is a real constant. The notations $(A \geq c)$, $(A_{m,n} < c)$, and so forth, will have similar meanings.

Lemma 2.74. *Let Ω and Γ be subsets of N_k^2, where $k \in Z$. Then*

$$\mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma). \tag{2.402}$$

Furthermore, if Ω and Γ are disjoint, then

$$\mu_*(\Omega) + \mu_*(\Gamma) \leq \mu_*(\Omega + \Gamma) \leq \mu_*(\Omega) + \mu^*(\Gamma) \leq \mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma), \tag{2.403}$$

so that

$$\mu_*(\Omega) + \mu^*(N_k^2 \setminus \Omega) = 1. \tag{2.404}$$

Proof. If Ω and Γ are disjoint, then $(\Omega + \Gamma)^{(m,n)} = \Omega^{(m,n)} + \Gamma^{(m,n)}$ so that

$$\begin{aligned} \mu^*(\Omega + \Gamma) &= \limsup_{m,n\to\infty} \frac{|\Omega^{(m,n)} + \Gamma^{(m,n)}|}{mn} \\ &\geq \limsup_{m,n\to\infty} \frac{|\Gamma^{(m,n)}|}{mn} + \liminf_{m,n\to\infty} \frac{|\Omega^{(m,n)}|}{mn} = \mu_*(\Omega) + \mu^*(\Gamma). \end{aligned} \tag{2.405}$$

The other cases are similarly proved. As an immediate consequence, we have

$$1 = \mu_*(N_k^2) \leq \mu_*(\Omega) + \mu^*(N_k^2 \setminus \Omega) \leq \mu^*(N_k^2) = 1. \tag{2.406}$$

Hence

$$\mu_*(\Omega) + \mu^*\left(N_k^2 \setminus \Omega\right) = 1. \tag{2.407}$$

□

Lemma 2.75. *Let Ω and Γ be subsets of N_k^2. If $\mu_*(\Omega) + \mu^*(\Gamma) > 1$, then $\Omega \cap \Gamma$ is an infinite set.*

Proof. If $\Omega \cap \Gamma$ is finite, then $\mu^*(\Omega \cap \Gamma) = 0$ and in view of $\Omega \subseteq (N_k^2 \setminus \Gamma) + \Omega \cap \Gamma$, we have

$$\mu^*(\Omega) \le \mu^*\left(N_k^2 \setminus \Gamma\right) + \mu^*(\Omega \cap \Gamma) = \mu^*\left(N_k^2 \setminus \Gamma\right). \tag{2.408}$$

Thus by Lemma 2.74,

$$1 < \mu^*(\Omega) + \mu_*(\Gamma) \le \mu^*\left(N_k^2 \setminus \Gamma\right) + \mu_*(\Gamma) = 1, \tag{2.409}$$

which is a contradiction. □

Similarly, from Lemma 2.74, we have the following.

Lemma 2.76. *Let $\Omega \subset N_k^2$, α, β, θ, and δ be integers such that $\alpha \le \beta$ and $\theta \le \delta$. Then*

$$\begin{aligned} \mu^*\left(\sum_{i=\alpha}^{\beta}\sum_{j=\theta}^{\delta} X^i Y^j \Omega\right) &\le (\beta-\alpha+1)(\delta-\theta+1)\mu^*(\Omega), \\ \mu_*\left(\sum_{i=\alpha}^{\beta}\sum_{j=\theta}^{\delta} X^i Y^j \Omega\right) &\le (\beta-\alpha+1)(\delta-\theta+1)\mu_*(\Omega). \end{aligned} \tag{2.410}$$

Lemma 2.77. *Let k, m, and n be three positive integers, and let $\{A_{i,j}\}$ be a sequence such that $A_{i,j} > 0$ for $i \in \{m, m+1, \dots, m+k\}$ and $j \in \{n, n+1, \dots, n+k\}$. If $dA_{i,j} \ge aA_{i+1,j} + bA_{i,j+1}$ for $i \in \{m, m+1, \dots, m+k\}$ and $j \in \{n, n+1, \dots, n+k\}$, then*

$$d^k A_{m,n} \ge \sum_{i=0}^{k} a^{k-i} b^i C_k^i A_{m+k-i,n+i}. \tag{2.411}$$

Proof. Obviously, (2.411) holds for $k = 1$. Assume that (2.411) holds for an integer $s \in \{1, 2, \dots, k-1\}$. Then in view of the following inequality:

$$\begin{aligned}
&\sum_{i=0}^{s} a^{s-i} b^i C_s^i \left(aA_{m+s+1-i,n+i} + bA_{m+s-i,n+i+1}\right) \\
&\quad \ge a^{s+1} A_{m+s+1,n} + \sum_{i=1}^{s} a^{s+1-i} b^i C_s^i A_{m+s+1-i,n+i} \\
&\qquad + \sum_{i=0}^{s-1} a^{s-i} b^{1+i} C_s^i A_{m+s-i,n+i+1} + b^{s+1} A_{m,n+s+1} \\
&\quad \ge a^{s+1} A_{m+s+1,n} + \sum_{i=1}^{s} a^{s+1-i} b^i \left(C_s^i + C_s^{i-1}\right) A_{m+s+1-i,n+i} \\
&\qquad + b^{s+1} A_{m,n+s+1} \\
&\quad = \sum_{i=0}^{s+1} a^{s+1-i} b^i C_{s+1}^i A_{m+s+1-i,n+i},
\end{aligned} \tag{2.412}$$

(2.411) holds for $s+1$. By induction, (2.411) holds. The proof is completed. □

Lemma 2.78. *Let k, m, and n be three positive integers such that $m \ge 2u$ and $n \ge 2v$. Assume that (2.398) has a solution $\{A_{i,j}\}$ such that $A_{i,j} > 0$ for $i \in \{m-2u, m-2u+1, \dots, m+k\}$ and $j \in \{n-2v, n-2v+1, \dots, n+k\}$, $p_s(i,j) \ge q_s \ge 0$ for $i \in \{m-u, m-u+1, \dots, m+k\}$ and $j \in \{n-v, n-v+1, \dots, n+k\}$, where q_s are real constants, $s = 1, 2, \dots, r$. Then*

$$\begin{aligned}
d^{k+1} A_{m,n} \ge{}& \sum_{i=0}^{k+1} a^{k+1-i} b^i C_{k+1}^i A_{m+k+1-i,n+i} \\
&+ (k+1) q \sum_{i=0}^{k} a^{k-i} b^i C_k^i A_{m+k-i-\alpha,n+i-\beta} \\
&+ q^2 \sum_{i=1}^{k} i d^{k-i} \sum_{j=0}^{i-1} a^{i-1-j} b^j C_{i-1}^j A_{m+i-1-j-2\alpha,n+j-2\beta},
\end{aligned} \tag{2.413}$$

where $\alpha = \min\{\sigma_1, \sigma_2, \dots, \sigma_r\}$ and $\beta = \min\{\tau_1, \tau_2, \dots, \tau_r\}$, and

$$q = \sum_{s=1}^{r} \frac{q_s a^{\sigma_s-\alpha} b^{\tau_s-\beta} C_{\sigma_s-\alpha+\tau_s-\beta}^{\tau_s-\beta}}{d^{\sigma_s-\alpha+\tau_s-\beta}}. \tag{2.414}$$

Proof. In view of (2.398), for any $i \in \{m-u, m-u+1, \dots, m+k\}$ and $j \in \{n-v, n-v+1, \dots, n+k\}$, we have

$$dA_{i,j} = aA_{i+1,j} + bA_{i,j+1} + \sum_{s=1}^{r} p_s(i,j) A_{i-\sigma_s, j-\tau_s} \geq aA_{i+1,j} + bA_{i,j+1}. \tag{2.415}$$

Then from Lemma 2.77, for any $i \in \{m, m+1, \dots, m+k\}$ and $j \in \{n, n+1, \dots, n+k\}$, we get

$$d^{\sigma_s+\tau_s-\alpha-\beta} A_{i-\sigma_s, j-\tau_s} \geq a^{\sigma_s-\alpha} b^{\tau_s-\beta} C_{\sigma_s+\tau_s-\alpha-\beta}^{\tau_s-\beta} A_{i-\alpha, j-\beta}, \quad s = 1, 2, \dots, r, \tag{2.416}$$

and so that

$$\begin{aligned} dA_{i,j} &\geq aA_{i+1,j} + bA_{i,j+1} + \left(\sum_{s=1}^{r} p_s(i,j) C_{\sigma_s+\tau_s-\alpha-\beta}^{\tau_s-\beta} \frac{a^{\sigma_s-\alpha} b^{\tau_s-\beta}}{d^{\sigma_s+\tau_s-\alpha-\beta}} \right) A_{i-\alpha, j-\beta} \\ &= aA_{i+1,j} + bA_{i,j+1} + q_{i,j} A_{i-\alpha, j-\beta}, \end{aligned} \tag{2.417}$$

where

$$q_{i,j} = \sum_{s=1}^{r} \frac{a^{\sigma_s-\alpha} b^{\tau_s-\beta} C_{\sigma_s-\alpha+\tau_s-\beta}^{\tau_s-\beta}}{d^{\sigma_s+\tau_s-\alpha-\beta}} p_s(i,j). \tag{2.418}$$

Obviously, $q_{i,j} \geq q$ for $i \in \{m-u, m-u+1, \dots, m+k\}$ and $j \in \{n-v, n-v+1, \dots, n+k\}$. Hence, from (2.417), we obtain

$$\begin{aligned} dA_{m,n} &\geq aA_{m+1,n} + bA_{m,n+1} + q_{m,n} A_{m-\alpha, n-\beta}, \\ dA_{m+1,n} &\geq aA_{m+2,n} + bA_{m+1,n+1} + q_{m+1,n} A_{m+1-\alpha, n-\beta}, \\ dA_{m,n+1} &\geq aA_{m+1,n+1} + bA_{m,n+2} + q_{m,n+1} A_{m-\alpha, n+1-\beta}, \\ dA_{m-\alpha,n-\beta} &\geq aA_{m+1-\alpha,n-\beta} + bA_{m-\alpha,n+1-\beta} + q_{m-\alpha,n-\beta} A_{m-2\alpha, n-2\beta}. \end{aligned} \tag{2.419}$$

Thus, from the above in equalities, we have

$$d^2A_{m,n} \geq a^2A_{m+2,n} + 2abA_{m+1,n+1} + b^2A_{m,n+2} + a(q_{m,n} + q_{m+1,n})A_{m+1-\alpha,n-\beta} + b(q_{m,n} + q_{m,n+1})A_{m-\alpha,n+1-\beta} + q_{m,n}q_{m-\alpha,n-\beta}A_{m-2\alpha,n-2\beta}. \tag{2.420}$$

Hence

$$d^2A_{m,n} \geq \sum_{i=0}^{2} a^{2-i}b^iC_2^iA_{m+2-i,n+i} + 2q\sum_{i=0}^{1} a^{1-i}b^iC_1^iA_{m+1-i-\alpha,n+i-\beta} + q^2\sum_{i=1}^{1} id^{1-i}\sum_{j=0}^{i-1} a^{i-1-j}b^jC_{i-1}^jA_{m+i-1-j-2\alpha,n+j-2\beta}. \tag{2.421}$$

Assume that (2.413) holds for a positive integer $s \in \{1,2,\dots,k\}$. Then from (2.412), (2.417), and the assumptions, we have

$$\begin{aligned}
d^{k+1}A_{m,n} \geq{} & \sum_{i=0}^{k} a^{k-i}b^iC_k^i\left(aA_{m+k+1-i,n+i} + bA_{m+k-i,n+1+i} + qA_{m+k-i-\alpha,n+i-\beta}\right) \\
& + kq\sum_{i=0}^{k-1} a^{k-1-i}b^iC_{k-1}^i \\
& \times \left(aA_{m+k-i-\alpha,n+i-\beta} + bA_{m+k-1-i-\alpha,n+1+i-\beta} + qA_{m+k-1-i-2\alpha,n+i-2\beta}\right) \\
& + q^2\sum_{i=1}^{k-1} id^{k-i}\sum_{j=0}^{i-1} a^{i-1-j}b^jC_{i-1}^jA_{m+i-1-j-2\alpha,n+j-2\beta} \\
\geq{} & \sum_{i=0}^{k+1} a^{k+1-i}b^iC_{k+1}^iA_{m+k+1-i,n+i} + q\sum_{i=0}^{k} a^{k-i}b^iC_k^iA_{m+k-i-\alpha,n+i-\beta} \\
& + kq\sum_{i=0}^{k} a^{k-i}b^iC_k^iA_{m+k-i-\alpha,n+i-\beta} + kq^2\sum_{i=0}^{k-1} a^{k-1-i}b^iC_{k-1}^iA_{m+k-1-i-2\alpha,n+i-2\beta} \\
& + q^2\sum_{i=1}^{k-1} id^{k-i}\sum_{j=0}^{i-1} a^{i-1-j}b^jC_{i-1}^jA_{m+i-1-j-2\alpha,n+j-2\beta} \\
={} & \sum_{i=0}^{k+1} a^{k+1-i}b^iC_{k+1}^iA_{m+k+1-i,n+i} + (k+1)q\sum_{i=0}^{k} a^{k-i}b^iC_k^iA_{m+k-i-\alpha,n+i-\beta} \\
& + q^2\sum_{i=1}^{k} id^{k-i}\sum_{j=0}^{i-1} a^{i-1-j}b^jC_{i-1}^jA_{m+i-1-j-2\alpha,n+j-2\beta}.
\end{aligned} \tag{2.422}$$

Hence (2.413) holds. The proof is completed. □

From Lemma 2.78, we can obtain the following corollaries.

Corollary 2.79. *Assume that* $\alpha > 0$ *and* $\beta > 0$. *Further, assume that for integers* $m \geq 3u$ *and* $n \geq 3v$, (2.398) *has a solution* $\{A_{i,j}\}$ *such that* $A_{i,j} > 0$ *for* $i \in \{m-3u, m-3u+1, \dots, m+v\}$ *and* $j \in \{n-3v, n-3v+1, \dots, n+u\}$, $p_s(i,j) \geq q_s \geq 0$ *for* $i \in \{m-2u, m-2u+1, \dots, m+v\}$, $j \in \{n-2v, n-2v+1, \dots, n+u\}$ *and* $s = 1, 2, \dots, r$, *where q is defined in* (2.414). *Then*

$$\left(d^{\alpha+\beta} - q(\alpha+\beta)C_{\alpha+\beta}^{\beta}\left(\frac{a^{\alpha}b^{\beta}}{d}\right)\right)A_{m-\alpha,n-\beta} \geq a^{\alpha}b^{\beta}C_{\alpha+\beta}^{\beta}A_{m,n}. \tag{2.423}$$

Proof. From (2.398), for $i \in \{m-2u, m-2u+1, \dots, m+v\}$ and $j \in \{n-2v, n-2v+1, \dots, n+u\}$, we have

$$dA_{i-1,j} \geq aA_{i,j}, \qquad dA_{i,j-1} \geq bA_{i,j}. \tag{2.424}$$

In view of Lemma 2.78 and the equality $C_k^i + C_k^{i-1} = C_{k+1}^i$, we have

$$\begin{aligned}
d^{\alpha+\beta}A_{m-\alpha,n-\beta} &\geq \sum_{i=0}^{\alpha+\beta} a^{\alpha+\beta-i}b^{i}C_{\alpha+\beta}^{i}A_{m+\beta-i,n-\beta+i} \\
&\quad + q(\alpha+\beta)\sum_{i=0}^{\alpha+\beta-1} a^{\alpha+\beta-1-i}b^{i}C_{\alpha+\beta-1}^{i}A_{m+\beta-1-i-\alpha,n+i-2\beta} \\
&\geq a^{\alpha}b^{\beta}C_{\alpha+\beta}^{\beta}A_{m,n} + (\alpha+\beta)qa^{\alpha-1}b^{\beta}C_{\alpha+\beta-1}^{\beta}A_{m-\alpha-1,n-\beta} \\
&\quad + (\alpha+\beta)qa^{\alpha}b^{\beta-1}C_{\alpha+\beta-1}^{\beta-1}A_{m-\alpha,n-\beta-1} \\
&\geq a^{\alpha}b^{\beta}C_{\alpha+\beta}^{\beta}A_{m,n} + (\alpha+\beta)q\left(\frac{a^{\alpha}b^{\beta}}{d}\right)C_{\alpha+\beta}^{\beta}A_{m-\alpha,n-\beta}.
\end{aligned} \tag{2.425}$$

Hence (2.423) holds. The proof is completed. □

Corollary 2.80. *Assume that* $\alpha > 0$ *and* $\beta > 0$. *Further assume that for integers* $m \geq 2u$ *and* $n \geq 2v$, (2.398) *has a solution* $\{A_{i,j}\}$ *such that* $A_{i,j} > 0$ *for* $i \in \{m-2u, m-2u+1, \dots, m+u+v+1\}$ *andj* $\in \{n-2v, n-2v+1, \dots, n+u+v+1\}$, *and* $p_s(i,j) \geq q_s \geq 0$ *for* $i \in \{m-u, m-u+1, \dots, m+u+v\}$, $j \in \{n-v, n-v+1, \dots, n+u+v\}$ *and* $s = 1,2,\dots,r$. *Let* q *be defined in* (2.414). *Then*

$$\left(d^{\alpha+\beta} - qd^{-1}a^{\alpha}b^{\beta}(1+\beta)C_{\alpha+\beta}^{\beta}\right)A_{m+1,n} \geq (\alpha+\beta)qa^{\alpha-1}b^{\beta}C_{\alpha+\beta-1}^{\beta}A_{m,n}, \tag{2.426}$$

$$\left(d^{\alpha+\beta} - qd^{-1}a^{\alpha}b^{\beta}(1+\alpha)C_{\alpha+\beta}^{\beta}\right)A_{m,n+1} \geq (\alpha+\beta)qa^{\alpha}b^{\beta-1}C_{\alpha+\beta-1}^{\beta-1}A_{m,n}. \tag{2.427}$$

Proof. From (2.398), we have $dA_{m+1,n-1} \geq bA_{m+1,n}$. From (2.417), for any $i \in \{m, m+1, \dots, m+u+v\}$ and $j \in \{n, n+1, \dots, n+u+v\}$, we obtain

$$dA_{i,j} \geq qA_{i-\alpha,j-\beta}. \tag{2.428}$$

In view of Lemma 2.78, we get

$$\begin{aligned} d^{\alpha+\beta}A_{m+1,n} \geq & \sum_{i=0}^{\alpha+\beta} a^{\alpha+\beta-i}b^{i}C_{\alpha+\beta}^{i}A_{m+1+\alpha+\beta-i,n+i} \\ & + q(\alpha+\beta)\sum_{i=0}^{\alpha+\beta-1} a^{\alpha+\beta-1-i}b^{i}C_{\alpha+\beta-1}^{i}A_{m+\alpha+\beta-i-\alpha,n+i-\beta} \\ \geq & \, a^{\alpha}b^{\beta}C_{\alpha+\beta}^{\beta}A_{m+1+\alpha,n+\beta} + (\alpha+\beta)qa^{\alpha-1}b^{\beta}C_{\alpha+\beta-1}^{\beta}A_{m,n} \\ & + (\alpha+\beta)qa^{\alpha}b^{\beta-1}C_{\alpha+\beta-1}^{\beta-1}A_{m+1,n-1} \\ \geq & \, qd^{-1}a^{\alpha}b^{\beta}C_{\alpha+\beta}^{\beta}A_{m+1,n} + (\alpha+\beta)qa^{\alpha-1}b^{\beta}C_{\alpha+\beta-1}^{\beta}A_{m,n} \\ & + qd^{-1}(\alpha+\beta)a^{\alpha}b^{\beta}C_{\alpha+\beta-1}^{\beta-1}A_{m+1,n} \\ = & \, qd^{-1}a^{\alpha}b^{\beta}(1+\beta)C_{\alpha+\beta}^{\beta}A_{m+1,n} + (\alpha+\beta)qa^{\alpha-1}b^{\beta}C_{\alpha+\beta-1}^{\beta}A_{m,n}. \end{aligned} \tag{2.429}$$

Hence (2.426) holds. Similarly, (2.427) holds. The proof is completed. □

Corollary 2.81. *Assume that* $\alpha > 0$ *and* $\beta > 0$. *Further, assume that for integers* $m \geq 2u+v$ *and* $n \geq 2v+u$, (2.398) *has a solution* $\{A_{i,j}\}$ *such that* $A_{i,j} > 0$ *for* $i \in \{m-2u-v, m-2u-v+1, \dots, m+2u+v+1\}$ *and* $j \in \{n-2v-u, n-2v-u+1, \dots, n+u+2v+1\}$, *and* $p_s(i,j) \geq q_s \geq 0$ *for* $i \in \{m-u-v, m-u-v+1, \dots, m+2u+v\}$, $j \in \{n-v-u, n-v-u+1, \dots, n+u+2v\}$ *and* $s = 1,2,\dots,r$.

Then

$$d^{\alpha+\beta+1}A_{m-h,n+h} \geq (\alpha+\beta+1)qa^{\alpha+h}b^{\beta-h}C_{\alpha+\beta}^{\beta-h}A_{m,n}, \tag{2.430}$$

where q is defined in (2.414) *and* $-\alpha \leq h \leq \beta$.

Proof. For any $-\alpha \leq h \leq \beta$, from Lemma 2.78, we get

$$\begin{aligned} d^{\alpha+\beta+1}A_{m-h,n+h} &\geq (\alpha+\beta+1)q\sum_{i=0}^{\alpha+\beta} a^{\alpha+\beta-i}b^{i}C_{\alpha+\beta}^{i}A_{m-h+\alpha+\beta-i-\alpha,n+h+i-\beta} \\ &\geq (\alpha+\beta+1)qa^{\alpha+h}b^{\beta-h}C_{\alpha+\beta}^{\beta-h}A_{m,n}. \end{aligned} \tag{2.431}$$

Hence (2.430) holds. The proof is completed. □

Theorem 2.82. *Assume that $\alpha > 0$ and $\beta > 0$. Further assume that there exist non-negative constants $q_i \geq 0$, $\theta_i \geq 0$, and $\omega \in [0,1]$ such that $\mu^*\{p_{m,n}^{(i)} < q_i\} = \theta_i \geq 0$, $i = 1,2,\ldots,r$,*

$$\begin{aligned} d^{\alpha+\beta+1} \leq q^2\Big\{&Dd^{-1}a^{\alpha+1}b^{\beta}C_{\alpha+\beta+1}^{\beta} + \overline{D}d^{-1}a^{\alpha}b^{\beta+1}C_{\alpha+\beta+1}^{\beta+1} \\ &+ (\alpha+\beta+1)^2Ea^{2\alpha}b^{2\beta}C_{2\alpha+2\beta}^{2\beta} + d^{-1}(\alpha+\beta)Ba^{\alpha}b^{\beta}C_{\alpha+\beta}^{\beta}\Big\}, \\ (4u+2v+1)&(2u+4v+1)(\theta_1+\theta_2+\cdots+\theta_r) + (5u+2v+2)(2u+5v+2)\omega < 1, \end{aligned} \tag{2.432}$$

where q is defined in (2.414) *and*

$$\begin{aligned} B &= \frac{a^{\alpha}b^{\beta}C_{\alpha+\beta}^{\beta}}{(d^{\alpha+\beta} - (q/d)(\alpha+\beta)a^{\alpha}b^{\beta}C_{\alpha+\beta}^{\beta})}, \\ E &= \frac{1}{d^{\alpha+\beta+1}}, \\ D &= \frac{(\alpha+\beta)a^{\alpha-1}b^{\beta}C_{\alpha+\beta-1}^{\beta}}{\{d^{\alpha+\beta} - qd^{-1}a^{\alpha}b^{\beta}(1+\beta)C_{\alpha+\beta}^{\beta}\}}, \\ \overline{D} &= \frac{(\alpha+\beta)a^{\alpha}b^{\beta-1}C_{\alpha+\beta-1}^{\beta-1}}{\{d^{\alpha+\beta} - qd^{-1}a^{\alpha}b^{\beta}(1+\alpha)C_{\alpha+\beta}^{\beta}\}}. \end{aligned} \tag{2.433}$$

Then every solution of (2.398) *is frequently oscillatory of lower degree ω.*

Proof. Suppose to the contrary, let $A = \{A_{m,n}\}$ be a frequently positive solution of (2.398) such that $\mu_*\{A \le 0\} \le \omega$. In view of Lemmas 2.74 and 2.76, we have

$$\begin{aligned}
&\mu_*\left\{N_0^2 \Big\backslash \sum_{s=1}^{r} \sum_{i=-2u-v}^{2u+v} \sum_{j=-u-2v}^{u+2v} X^i Y^j\left(p_s(m,n) < q_s\right)\right\} \\
&\quad + \mu^*\left\{N_0^2 \Big\backslash \sum_{i=-2u-v-1}^{3u+v} \sum_{j=-u-2v-1}^{u+3v} X^i Y^j(A \le 0)\right\} \\
&\quad = 2 - \mu^*\left\{\sum_{s=1}^{r} \sum_{i=-2u-v}^{2u+v} \sum_{j=-u-2v}^{u+2v} X^i Y^j\left(p_s(m,n) < q_s\right)\right\} \\
&\qquad - \mu_*\left\{\sum_{i=-2u-v-1}^{3u+v} \sum_{j=-u-2v-1}^{u+3v} X^i Y^j(A \le 0)\right\} \\
&\quad \ge 2 - (4u+2v+1)(2u+4v+1)\left(\theta_1 + \cdots + \theta_u\right) \\
&\qquad - (5u+2v+2)(2u+5v+2)\omega > 1.
\end{aligned} \tag{2.434}$$

Hence by Lemma 2.75, the intersection

$$\begin{aligned}
&\left\{N_0^2 \Big\backslash \sum_{s=1}^{r} \sum_{i=-2u-v}^{2u+v} \sum_{j=-u-2v}^{u+2v} X^i Y^j\left(p_s(m,n) < q_s\right)\right\} \\
&\quad \bigcap \left\{N_0^2 \Big\backslash \sum_{i=-2u-v-1}^{3u+v} \sum_{j=-u-2v-1}^{u+3v} X^i Y^j(A \le 0)\right\}
\end{aligned} \tag{2.435}$$

is an infinite subset of N_0^2, which together with (2.401) implies that there exists a lattice point (m,n) such that $A_{i,j} > 0$ for $i \in \{m-3u-v, m-3u-v+1, \dots, m+2u+v+1\}$ and $j \in \{n-3v-u, n-3v-u+1, \dots, n+u+2v+1\}$, and $p_s(i,j) \ge q_s$ for $i \in \{m-2u-v, m-2u-v+1, \dots, m+2u+v\}$ and $j \in \{n-2v-u, n-2v-u+1, \dots, n+u+2v\}$, $s = 1,2,\dots,r$. If $\alpha \ge \beta$, then from (2.417) and Corollaries 2.79–2.81, we get

$$\begin{gathered}
A_{m-h,n+h} \ge E(\alpha+\beta+1)qa^{\alpha+h}b^{\beta-h}C_{\alpha+\beta}^{\beta-h}A_{m,n} \quad \text{for } -\alpha \le h \le \beta, \\
dA_{m+\alpha+1,n+\beta} \ge qA_{m+1,n}, \qquad dA_{m+\alpha,n+\beta+1} \ge qA_{m,n+1}, \\
A_{m+1,n} \ge DqA_{m,n}, \qquad A_{m,n+1} \ge \overline{D}qA_{m,n}, \\
A_{m-\alpha-1,n-\beta} \ge BA_{m-1,n}, \qquad dA_{m-1,n} \ge aA_{m,n}, \qquad dA_{m,n-1} \ge bA_{m,n}.
\end{gathered} \tag{2.436}$$

Hence, from Lemma 2.78,

$$\begin{aligned}
d^{\alpha+\beta+1}A_{m,n} \geq & \sum_{i=0}^{\alpha+\beta+1} a^{\alpha+\beta+1-i}b^i C_{\alpha+\beta+1}^i A_{m+\alpha+\beta+1-i,n+i} \\
& + (\alpha+\beta+1)q\sum_{i=0}^{\alpha+\beta} a^{\alpha+\beta-i}b^i C_{\alpha+\beta}^i A_{m+\alpha+\beta-i-\alpha,n+i-\beta} \\
& + q^2\sum_{i=1}^{\alpha+\beta} id^{\alpha+\beta-i}\sum_{j=0}^{i-1} a^{i-1-j}b^j C_{i-1}^j A_{m+i-1-j-2\alpha,n+j-2\beta} \\
& > a^{\alpha+1}b^\beta C_{\alpha+\beta+1}^\beta A_{m+\alpha+1,n+\beta} + a^\alpha b^{\beta+1}C_{\alpha+\beta+1}^{\beta+1}A_{m+\alpha,n+\beta+1} \\
& + q(\alpha+\beta+1)\sum_{i=0}^{2\beta} a^{\alpha+\beta-i}b^i C_{\alpha+\beta}^i A_{m+\beta-i,n+i-\beta} \\
& + q^2(\alpha+\beta)a^{\alpha-1}b^\beta C_{\alpha+\beta-1}^\beta A_{m-\alpha-1,n-\beta} \\
& + q^2(\alpha+\beta)a^\alpha b^{\beta-1}C_{\alpha+\beta-1}^{\beta-1}A_{m-\alpha,n-\beta-1} \\
& \geq \left(q^2Dd^{-1}a^{\alpha+1}b^\beta C_{\alpha+\beta+1}^\beta + q^2\overline{D}d^{-1}a^\alpha b^{\beta+1}C_{\alpha+\beta+1}^{\beta+1}\right)A_{m,n} \\
& + (\alpha+\beta+1)^2q^2Ea^{2\alpha}b^{2\beta}\left(\sum_{i=0}^{2\beta} C_{\alpha+\beta}^i C_{\alpha+\beta}^{2\beta-i}\right)A_{m,n} \\
& + q^2d^{-1}B(\alpha+\beta)a^\alpha b^\beta C_{\alpha+\beta}^\beta A_{m,n}.
\end{aligned} \tag{2.437}$$

In view of the equality $\sum_{i=0}^{2\beta} C_{\alpha+\beta}^i C_{\alpha+\beta}^{2\beta-i} = C_{2\alpha+2\beta}^{2\beta}$, we have

$$\begin{aligned}
d^{\alpha+\beta+1} > q^2\Big\{ & Dd^{-1}a^{\alpha+1}b^\beta C_{\alpha+\beta+1}^\beta + \overline{D}d^{-1}a^\alpha b^{\beta+1}C_{\alpha+\beta+1}^{\beta+1} \\
& + (\alpha+\beta+1)^2Ea^{2\alpha}b^{2\beta}C_{2\alpha+2\beta}^{2\beta} + d^{-1}B(\alpha+\beta)a^\alpha b^\beta C_{\alpha+\beta}^\beta\Big\},
\end{aligned} \tag{2.438}$$

which is contrary to (2.432).

If $\alpha < \beta$, similar to the above proof, we have

$$\begin{aligned}
d^{\alpha+\beta+1}A_{m,n} > & \left(q^2Dd^{-1}a^{\alpha+1}b^\beta C_{\alpha+\beta+1}^\beta + q^2\overline{D}d^{-1}a^\alpha b^{\beta+1}C_{\alpha+\beta+1}^{\beta+1}\right)A_{m,n} \\
& + (\alpha+\beta+1)^2q^2Ea^{2\alpha}b^{2\beta}\left(\sum_{i=\beta-\alpha}^{\alpha+\beta} C_{\alpha+\beta}^i C_{\alpha+\beta}^{2\beta-i}\right)A_{m,n} \\
& + q^2d^{-1}B(\alpha+\beta)a^\alpha b^\beta C_{\alpha+\beta}^\beta A_{m,n}.
\end{aligned} \tag{2.439}$$

In view of the equality

$$\sum_{i=\beta-\alpha}^{\alpha+\beta} C_{\alpha+\beta}^{i} C_{\alpha+\beta}^{2\beta-i} = \sum_{i=\beta-\alpha}^{\alpha+\beta} C_{\alpha+\beta}^{\alpha+\beta-i} C_{\alpha+\beta}^{2\alpha-(\alpha+\beta-i)} = \sum_{i=0}^{2\alpha} C_{\alpha+\beta}^{i} C_{\alpha+\beta}^{2\alpha-i} = C_{2\alpha+2\beta}^{2\alpha} = C_{2\alpha+2\beta}^{2\beta}, \tag{2.440}$$

we also obtain a contradiction to (2.432). The proof is completed. □

Corollary 2.83. *Assume that* $\alpha > 0$ *and* $\beta > 0$. *Further assume that there exist positive constants* $q_i \geq 0$ *such that* $\mu\{p_{m,n}^{(i)} < q_i\} = 0$, $i = 1, 2, \ldots, r$, *and*

$$q \geq \frac{d^{\alpha+\beta+1}}{(\alpha+\beta+1)a^{\alpha}b^{\beta}\sqrt{C_{2\alpha+2\beta}^{2\beta}}}. \tag{2.441}$$

Then every solution of (2.398) *is frequently oscillatory of lower degree* ω *(and hence oscillatory), where* $\omega \in [0, [(5u+2v+2)(2u+5v+2)]^{-1})$.

In fact, in view of $d^{\alpha+\beta+1} = Ed^{2(\alpha+\beta+1)}$, from (2.441), then

$$d^{\alpha+\beta+1} \leq q^2(\alpha+\beta+1)^2 E a^{2\alpha} b^{2\beta} C_{2\alpha+2\beta}^{2\beta}. \tag{2.442}$$

Hence (2.432) holds. By Theorem 2.82, Corollary 2.83 follows.

Similarly, from Theorem 2.82, it is easy to obtain the following corollaries.

Corollary 2.84. *Assume that* $\sigma > 0$ *and* $\tau > 0$, *and*

$$\mu\{p_{m,n} > \Theta\} = 1, \tag{2.443}$$

where

$$\Theta = \left\{(\sigma+\tau)\left[C_{\sigma+\tau-1}^{\tau} C_{\sigma+\tau+1}^{\tau} + C_{\sigma+\tau-1}^{\tau-1} C_{\sigma+\tau+1}^{\tau+1} + \left(C_{\sigma+\tau}^{\tau}\right)^2\right] + (\sigma+\tau+1)^2 C_{2\sigma+2\tau}^{2\tau}\right\}^{-1/2}. \tag{2.444}$$

Then every solution of equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n} A_{m-\sigma,n-\tau} = 0 \tag{2.445}$$

is frequently oscillatory of lower degree ω *(and hence oscillatory), where* $\omega \in [0, [(5u+2v+2)(2u+5v+2)]^{-1})$.

Corollary 2.85. *Assume that* $\sigma > 0$ *and* $\tau > 0$, *and*

$$\liminf_{m,n\to\infty} p_{m,n} > \Theta, \tag{2.446}$$

where Θ *is defined above. Then every solution of* (2.445) *is frequently oscillatory (and hence oscillatory).*

In the sequel, we give two examples to illustrate the above results.

Example 2.86. Consider the partial difference equation with two delays of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-1,n-2} + q_{m,n}A_{m-2,n-1} = 0, \tag{2.447}$$

where $p_{m,n} = -1$ and $q_{m,n} = -1$ for $(m,n) \in S = \{(i,j) \mid i = 2^s,\ j = 2^t,\ s,t = 0,1,2,\dots\}$, and $p_{m,n} = 0.05$ and $q_{m,n} = 0.07$ for any $(m,n) \notin S$. Let $a = b = d = 1$, $r = 2$, $\sigma_1 = 1$, $\tau_1 = 2$ and $\sigma_2 = 2$ and $\tau_2 = 1$, then $\alpha = 1$ and $\beta = 1$. It is obvious that $\mu\{p_{m,n} \geq 0.05 = q_1\} = 1$ and $\mu\{q_{m,n} \geq 0.07 = q_2\} = 1$, $E = 1$, and

$$q = \sum_{s=1}^{r} \frac{q_s a^{\sigma_s-\alpha} b^{\tau_s-\beta} C_{\sigma_s-\alpha+\tau_s-\beta}^{\tau_s-\beta}}{d^{\sigma_s-\alpha+\tau_s-\beta}} = q_1 + q_2 = 0.12,$$

$$\begin{aligned} B &= \frac{a^\alpha b^\beta C_{\alpha+\beta}^{\beta}}{\left(d^{\alpha+\beta} - q(\alpha+\beta)d^{-1}a^\alpha b^\beta C_{\alpha+\beta}^{\beta}\right)} > 2, \\ D &= \frac{(\alpha+\beta)a^{\alpha-1} b^\beta C_{\alpha+\beta-1}^{\beta}}{\left(d^{\alpha+\beta} - qd^{-1}a^\alpha b^\beta(1+\beta)C_{\alpha+\beta}^{\beta}\right)} > 2, \\ \overline{D} &= \frac{(\alpha+\beta)a^{\alpha} b^{\beta-1} C_{\alpha+\beta-1}^{\beta-1}}{\left(d^{\alpha+\beta} - qd^{-1}a^\alpha b^\beta(1+\alpha)C_{\alpha+\beta}^{\beta}\right)} > 2. \end{aligned} \tag{2.448}$$

Obviously,

$$\frac{1}{\sqrt{D \times C_3^1 + \overline{D} \times C_3^2 + 9 \times C_4^2 + 2 \times B \times C_2^1}} < \frac{1}{\sqrt{6+6+54+8}} = \frac{1}{\sqrt{74}} < q. \tag{2.449}$$

Hence (2.432) holds with $u = v = 2$ and $\theta_1 = \theta_2 = 0$. By Theorem 2.82, every solution of (2.447) is frequently oscillatory of lower degree $\omega \in [0, 1/256]$ and hence oscillatory.

Example 2.87. Consider the partial difference equation of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-1,n-2} = 0, \tag{2.450}$$

where $p_{m,n} = 1/16$ for any $m,n = 0,1,2,\dots$. Let $\sigma = 1$ and $\tau = 2$. It is easy to see that $p_{m,n} = 1/16 = 0.0625$,

$$\frac{(\sigma+\tau)^{\sigma+\tau}}{(\sigma+\tau+1)^{\sigma+\tau+1}} = \frac{3^3}{4^4} \approx 0.1055, \qquad \Theta = \frac{1}{\sqrt{309}} \approx 0.0569. \tag{2.451}$$

Hence from Corollary 2.85, every solution of (2.450) is frequently oscillatory and hence oscillatory. This conclusion cannot be obtained from Corollary 2.18.

2.8. Linear PDEs with unbounded delays

In this section, we will consider the partial difference equation

$$A_{m+1,n} + a_{m,n}A_{m,n+1} - b_{m,n}A_{m,n} + p_{m,n}A_{\sigma(m),\tau(n)} = 0, \tag{2.452}$$

where $\{a_{m,n}\}$, $\{b_{m,n}\}$, and $\{p_{m,n}\}$ are three real double sequences, $m, n \in N_0$. For (2.452), we always assume that the following hypotheses, designated by (H), hold:

(i) σ and τ: $N \to Z$ are nondecreasing;
(ii) $\sigma(n) < n$ and $\tau(n) < n$ for all $n \in N$;
(iii) $\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \tau(n) = \infty$;
(iv) $a_{m,n} \geq a$ and $b_{m,n} \leq b$, $p_{m,n} \geq 0$ for all large m and n, where a and b are two positive constants.

For example, we see that $\sigma(m) = [m/2]$ and $\tau(n) = [n/2]$ satisfy condition (H), where $[\cdot]$ denotes the greatest integer. Hence, (2.452) includes partial difference equations with unbounded delay.

Lemma 2.88. *For $m \geq M$ and $n \geq N$, the following formal identity holds:*

$$\begin{aligned}
&\sum_{i=M}^{m}\sum_{j=N}^{n}\left(A_{i+1,j} + aA_{i,j+1} - bA_{i,j}\right)\\
&\quad = (1+a-b)\sum_{i=M+1}^{m}\sum_{j=N+1}^{n}A_{i,j} + \sum_{j=N+1}^{n}A_{m+1,j} + (a-b)\sum_{j=N+1}^{n}A_{M,j}\\
&\qquad + a\sum_{i=M}^{m}A_{i,n+1} + (1-b)\sum_{i=M+1}^{m}A_{i,N} + A_{m+1,N} - bA_{M,N}\\
&\quad = (1+a-b)\sum_{i=M+1}^{m}\sum_{j=N+1}^{n}A_{i,j} + a\sum_{i=M+1}^{m}A_{i,n+1} + (a-b)\sum_{j=N+1}^{n}A_{M,j}\\
&\qquad + \sum_{j=N}^{n}A_{m+1,j} + (1-b)\sum_{i=M+1}^{m}A_{i,N} + aA_{M,n+1} - bA_{M,N}.
\end{aligned} \tag{2.453}$$

Lemma 2.89. *Assume that (H) holds and $\{A_{i,j}\}$ is an eventually positive solution of (2.452) such that $A_{i,j} > 0$ and $p_{i,j} \geq 0$ for $i \geq \sigma(M)$ and $j \geq \tau(N)$, where M and N are two sufficiently large integers. Then for any integer $k \geq 0$ and $m \geq M$ and $n \geq N$,*

$$b^{k+1}A_{m,n} \geq \sum_{i=0}^{k+1} a^i C_{k+1}^i A_{m+k+1-i,n+i}. \tag{2.454}$$

Proof. In view of (2.452) and (H), for $m \geq M$ and $n \geq N$, we have

$$bA_{m,n} \geq A_{m+1,n} + aA_{m,n+1}, \tag{2.455}$$

and thus,

$$bA_{m+1,n} \geq A_{m+2,n} + aA_{m+1,n+1}, \qquad bA_{m,n+1} \geq A_{m+1,n+1} + aA_{m,n+2}. \tag{2.456}$$

Hence, from (2.455), we have

$$b^2 A_{m,n} \geq A_{m+2,n} + 2aA_{m+1,n+1} + a^2 A_{m,n+2} = \sum_{i=0}^{2} a^i C_2^i A_{m+2-i,n+i}. \tag{2.457}$$

Assume that for any positive integer $k \geq 1$,

$$b^k A_{m,n} \geq \sum_{i=0}^{k} a^i C_k^i A_{m+k-i,n+i}. \tag{2.458}$$

Then for $0 \leq i \leq k$, from (2.455) we have

$$A_{m+k+1-i,n+i} + aA_{m+k-i,n+1+i} \leq bA_{m+k-i,n+i}. \tag{2.459}$$

Combining the last two inequalities, we obtain

$$b^{k+1} A_{m,n} \geq \sum_{i=0}^{k} a^i C_k^i \left(A_{m+k+1-i,n+i} + aA_{m+k-i,n+1+i}\right). \tag{2.460}$$

Since

$$\begin{aligned}
&\sum_{i=0}^{k} a^i C_k^i \left(A_{m+k+1-i,n+i} + aA_{m+k-i,n+1+i}\right) \\
&\quad = A_{m+k+1,n} + \sum_{i=1}^{k} a^i C_k^i A_{m+k+1-i,n+i} \\
&\qquad + \sum_{i=0}^{k-1} a^{i+1} C_k^i A_{m+k-i,n+1+i} + a^{k+1} A_{m,n+k+1} \\
&\quad = A_{m+k+1,n} + \sum_{i=1}^{k} a^i \left(C_k^i + C_k^{i-1}\right) A_{m+k+1-i,n+i} \\
&\qquad + a^{k+1} A_{m,n+k+1} = \sum_{i=0}^{k+1} a^i C_{k+1}^i A_{m+k+1-i,n+i},
\end{aligned} \tag{2.461}$$

then we have

$$b^{k+1}A_{m,n} \geq \sum_{i=0}^{k+1} a^i C_{k+1}^i A_{m+k+1-i,n+i}. \tag{2.462}$$

The proof is completed by induction. □

Corollary 2.90. *Assume that (H) holds and* $\{A_{m,n}\}$ *is an eventually positive solution of* (2.452) *and* $a \geq b$, $b \leq 1$. *Then* $A_{m,n}$ *tends to zero as* $m, n \to \infty$.

Proof. Assume that $A_{m,n} > 0$ and $p_{m,n} \geq 0$ for $m \geq \sigma(M)$ and $n \geq \tau(N)$, where M and N are two positive integers. By means of Lemma 2.89, for all positive integers k and l,

$$b^{k+l}A_{M,N} \geq a^l C_{k+l}^l A_{M+k,N+l}. \tag{2.463}$$

Thus,

$$A_{M+k,N+l} \leq \frac{b^k A_{M,N}}{C_{k+l}^l}\left(\frac{b}{a}\right)^l \longrightarrow 0 \quad \text{as } k, l \longrightarrow \infty. \tag{2.464}$$

The proof is completed. □

By Lemma 2.89, it is easy to obtain the following corollary.

Corollary 2.91. *Assume that (H) holds and* $\{A_{m,n}\}$ *is an eventually positive solution of* (2.452) *so that* $A_{m,n} > 0$ *and* $p_{m,n} \geq 0$ *for* $m \geq \sigma^2(M)$ *and* $n \geq \tau^2(N)$. *Then*

$$b^{m-\sigma(m)+n-\tau(n)}A_{\sigma(m),\tau(n)} \geq a^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} A_{m,n} \tag{2.465}$$

for $m \geq M$ *and* $n \geq N$, *where* M *and* N *are two positive integers.*

Lemma 2.92. *Assume that (H) holds,* $a \geq b$, $b \leq 1$, *and* $\{A_{m,n}\}$ *is an eventually positive solution of* (2.452). *If there exists* $B > 0$ *such that for sufficiently large* M *and* N,

$$\sum_{i=M}^{m}\sum_{j=N}^{n} p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \geq B, \tag{2.466}$$

then

$$bA_{M,N} \geq A_{m+1,N} + BA_{\sigma(m),\tau(n)}, \tag{2.467}$$

$$bA_{M,N} \geq aA_{M,n+1} + BA_{\sigma(m),\tau(n)}. \tag{2.468}$$

Proof. In view of (2.452), (2.453), and Corollary 2.91, for sufficiently large M and N, we obtain

$$
\begin{aligned}
0 &\geq \sum_{i=M}^{m}\sum_{j=N}^{n}\left(A_{i+1,j}+aA_{i,j+1}-bA_{i,j}+p_{i,j}A_{\sigma(i),\tau(j)}\right)\\
&\geq \sum_{i=M}^{m}\sum_{j=N}^{n}p_{i,j}A_{\sigma(i),\tau(j)}+A_{m+1,N}-bA_{M,N}\\
&\geq \sum_{i=M}^{m}\sum_{j=N}^{n}p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)}C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)}A_{\sigma(m),\tau(n)}\\
&\quad + A_{m+1,N}-bA_{M,N}\\
&\geq BA_{\sigma(m),\tau(n)}+A_{m+1,N}-bA_{M,N}
\end{aligned}
\tag{2.469}
$$

and hence, inequality (2.467) holds. On the other hand, from (2.452) and (2.453) we find

$$
0 \geq \sum_{i=M}^{m}\sum_{j=N}^{n}p_{i,j}A_{\sigma(i),\tau(j)}+aA_{M,n+1}-bA_{M,N}. \tag{2.470}
$$

By a similar argument as above, we obtain (2.468). The proof is completed. □

Lemma 2.93. *Assume that (H) holds, $a \geq b$, $b \leq 1$, and $\{A_{m,n}\}$ is an eventually positive solution of* (2.452), *and for all large m and n, then*

$$
\sum_{i=\sigma(m)}^{m-1}\sum_{j=\tau(n)}^{n-1}p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)}C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \geq B > 0. \tag{2.471}
$$

Then for all large m and n, then

$$
\frac{A_{\sigma^2(m),\tau(n)}}{A_{\sigma(m),n}} \leq \left(\frac{2b}{B}\right)^4, \tag{2.472}
$$

where $\sigma^0(m) = m$ and $\sigma^k(m) = \sigma(\sigma^{k-1}(m))$, $k = 1,2,\ldots$.

Proof. In view of (2.471), for large m and n, there exists an integer $\overline{m}$ such that $m \in \{\sigma(\overline{m}),\sigma(\overline{m})+1,\ldots,\overline{m}\}$ and

$$
\begin{aligned}
&\sum_{i=\sigma(\overline{m})}^{m}\sum_{j=\tau(n)}^{n}p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)}C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \geq \frac{B}{2},\\
&\sum_{i=m}^{\overline{m}}\sum_{j=\tau(n)}^{n}p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)}C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \geq \frac{B}{2}.
\end{aligned}
\tag{2.473}
$$

By Lemma 2.92, we have

$$bA_{\sigma(\overline{m}),\tau(n)} \geq A_{m+1,\tau(n)} + \frac{B}{2}A_{\sigma(m),\tau(n)},$$
$$bA_{m,\tau(n)} \geq A_{\overline{m}+1,\tau(n)} + \frac{B}{2}A_{\sigma(\overline{m}),\tau(n)}. \tag{2.474}$$

Hence, $A_{m,\tau(n)} \geq (B/2b)^2 A_{\sigma(m),\tau(n)}$ for large m and n. Similarly, $A_{\sigma(m),n} \geq (B/2b)^2 A_{\sigma(m),\tau(n)}$ for large m and n. Thus, for all large m and n, we have

$$\frac{A_{\sigma^2(m),\tau(n)}}{A_{\sigma(m),n}} = \frac{A_{\sigma^2(m),\tau(n)}}{A_{\sigma(m),\tau(n)}} \cdot \frac{A_{\sigma(m),\tau(n)}}{A_{\sigma(m),n}} \leq \left(\frac{2b}{B}\right)^4. \tag{2.475}$$

The proof is completed. □

Theorem 2.94. *Assume that (H) holds and*

$$\limsup_{m,n\to\infty} p_{m,n}\frac{1}{b^{m-\sigma(m)+1}}\left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} > 1. \tag{2.476}$$

Then every solution of (2.452) *is oscillatory.*

Proof. Suppose to the contrary, there is an eventually positive solution $\{A_{m,n}\}$ of (2.452) such that $A_{m,n} > 0$ and $p_{m,n} \geq 0$ for $m \geq \sigma^2(M)$ and $n \geq \tau^2(N)$, where M and N are two positive integers. By means of Corollary 2.91, we have for $m \geq M$ and $n \geq N$,

$$A_{m+1,n} + aA_{m,n+1} - bA_{m,n} + p_{m,n}\frac{1}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} A_{m,n} \leq 0, \tag{2.477}$$

that is,

$$p_{m,n}\frac{1}{b^{m-\sigma(m)+1}}\left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} \leq 1 \quad \text{for } m \geq M,\ n \geq N, \tag{2.478}$$

which is a contradiction to (2.476). The proof is completed. □

The following two corollaries can be easily derived from Theorem 2.94, and their proofs are thus omitted.

Corollary 2.95. *Assume that (H) holds, $a > b$, and $b \leq 1$. If either* $\lim_{m\to\infty}(m - \sigma(m)) = \infty$ *or* $\lim_{n\to\infty}(n - \tau(n)) = \infty$ *holds, and* $\limsup_{m,n\to\infty} p_{m,n} > 0$, *then every solution of* (2.452) *is oscillatory.*

Corollary 2.96. *Assume that (H) holds, $\sigma(m) = m - \sigma$, and $\tau(n) = n - \tau$, where σ and τ are two positive integers. If*

$$\limsup_{m,n\to\infty} p_{m,n} > \frac{b^{\sigma+1}}{C^{\tau}_{\sigma+\tau}}\left(\frac{b}{a}\right)^{\tau}, \tag{2.479}$$

then every solution of (2.452) *is oscillatory.*

If (2.476) does not hold, then we have the following result.

Theorem 2.97. *Assume that (H) holds and*

$$\limsup_{m,n\to\infty} \frac{1}{d_{m,n}} \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C^{\tau(n)-\tau(j)}_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)} > 1. \tag{2.480}$$

Then every solution of (2.452) *is oscillatory, where*

$$d_{m,n} = \begin{cases} b, & a \geq b,\ b \leq 1, \\ b^{m-\sigma(m)+1}, & a \geq b,\ b \geq 1, \\ b\left(\dfrac{b}{a}\right)^{n-\tau(n)}, & a \leq b,\ b \leq 1, \\ b\left[\left(\dfrac{b}{a}\right)^{n-\tau(n)} - 1 + b^{m-\sigma(m)}\right], & a \leq b,\ b \geq 1,\ b-a \leq 1, \\ b\left[\left(\dfrac{b}{a}\right)^{n-\tau(n)} - 1 + b^{m-\sigma(m)}\right] - (1+a-b) & \\ \quad \times \displaystyle\sum_{i=\sigma(m)+1}^{m} \sum_{j=\tau(n)+1}^{n} \frac{b^{i-\sigma(m)}}{C^{j-\tau(n)}_{i-\sigma(m)+j-\tau(n)}}\left(\frac{b}{a}\right)^{j-\tau(n)}, & a \leq b,\ b \geq 1,\ b-a \geq 1. \end{cases} \tag{2.481}$$

Proof. Assume that there exists an eventually positive solution $\{A_{m,n}\}$ of (2.452) such that $A_{m,n} > 0$ and $p_{m,n} \geq 0$ for $m \geq \sigma^2(M)$ and $n \geq \tau^2(N)$, where M and N are two sufficiently large positive integers. Then in view of (2.452), Lemmas 2.88 and 2.89, for $m \geq M$ and $n \geq N$, we have

$$A_{m+1,n} + aA_{m,n+1} - bA_{m,n} + p_{m,n}A_{\sigma(m),\tau(n)} \leq 0, \tag{2.482}$$

and thus,

$$
\begin{aligned}
0 &\geq \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} \left(A_{i+1,j} + aA_{i,j+1} - bA_{i,j} + p_{i,j}A_{\sigma(i),\tau(j)}\right) \\
&= \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j}A_{\sigma(i),\tau(j)} \\
&\quad + (1+a-b) \sum_{i=\sigma(m)+1}^{m} \sum_{j=\tau(n)+1}^{n} A_{i,j} + \sum_{j=\tau(n)+1}^{n} A_{m+1,j} \\
&\quad + a \sum_{i=\sigma(m)}^{m} A_{i,n+1} + (a-b) \sum_{j=\tau(n)+1}^{n} A_{\sigma(m),j} \\
&\quad + (1-b) \sum_{i=\sigma(m)+1}^{m} A_{i,\tau(n)} + A_{m+1,\tau(n)} - bA_{\sigma(m),\tau(n)}.
\end{aligned} \tag{2.483}
$$

Case A ($a \geq b$, $b \leq 1$). Inequality (2.483) provides

$$
0 \geq \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j}A_{\sigma(i),\tau(j)} - bA_{\sigma(m),\tau(n)}. \tag{2.484}
$$

By Lemma 2.89, we have for $\sigma(m) \leq i \leq m$ and $\tau(n) \leq j \leq n$,

$$
b^{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}A_{\sigma(i),\tau(j)} \geq a^{\tau(n)-\tau(j)}C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)}A_{\sigma(m),\tau(n)}. \tag{2.485}
$$

It follows that

$$
0 \geq \left\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} - b \right\} A_{\sigma(m),\tau(n)}, \tag{2.486}
$$

or equivalently,

$$
\sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j}\frac{1}{b^{\sigma(m)-\sigma(i)}}\left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \leq b, \tag{2.487}
$$

which is a contradiction to (2.480).

Case B ($a \ge b$, $b \ge 1$). It follows from (2.483) and Lemma 2.89 that

$$\begin{aligned}
0 &\ge \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} A_{\sigma(i),\tau(j)} + (1-b) \sum_{i=\sigma(m)+1}^{m} A_{i,\tau(n)} - b A_{\sigma(m),\tau(n)} \\
&\ge \Bigg\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \\
&\qquad + (1-b) \sum_{i=\sigma(m)+1}^{m} b^{i-\sigma(m)} - b \Bigg\} A_{\sigma(m),\tau(n)}.
\end{aligned} \tag{2.488}$$

A similar contradiction as in Case A is thus obtained.

Case C ($a \le b$, $b \le 1$). Since $b \le 1$, we have $1 + a - b > 0$. Consequently, from (2.483) and Lemma 2.89, we find

$$\begin{aligned}
0 &\ge \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} A_{\sigma(i),\tau(j)} + (a-b) \sum_{j=\tau(n)+1}^{n} A_{\sigma(m),j} - b A_{\sigma(m),\tau(n)} \\
&\ge \Bigg\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \\
&\qquad + (a-b) \sum_{j=\tau(n)+1}^{n} \left(\frac{b}{a}\right)^{j-\tau(n)} - b \Bigg\} A_{\sigma(m),\tau(n)},
\end{aligned} \tag{2.489}$$

which contradicts to (2.480).

Case D ($a \le b$, $b \ge 1$, $b - a \le 1$). Since $b - a \le 1$, then $1 + a - b \ge 0$. Hence, it follows from (2.483) that

$$\begin{aligned}
0 &\ge \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} A_{\sigma(i),\tau(j)} \\
&\quad + (a-b) \sum_{j=\tau(n)+1}^{n} A_{\sigma(m),j} + (1-b) \sum_{i=\sigma(m)+1}^{m} A_{i,\tau(n)} - b A_{\sigma(m),\tau(n)} \\
&\ge \Bigg\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \\
&\qquad + (a-b) \sum_{j=\tau(n)+1}^{n} \left(\frac{b}{a}\right)^{j-\tau(n)} + (1-b) \sum_{i=\sigma(m)+1}^{m} b^{i-\sigma(m)} - b \Bigg\} A_{\sigma(m),\tau(n)}.
\end{aligned} \tag{2.490}$$

The rest of the proof is similar to that of Cases A–C.

Case E ($a \le b$, $b \ge 1$, $b - a \ge 1$). Since $b - a \ge 1$, then $1 + a - b \le 0$. Therefore, from (2.483) and Lemma 2.89, we find

$$\begin{aligned}
0 \ge& \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} A_{\sigma(i),\tau(j)} + (1+a-b) \sum_{i=\sigma(m)+1}^{m} \sum_{j=\tau(n)+1}^{n} A_{i,j} \\
&+ (a-b) \sum_{j=\tau(n)+1}^{n} A_{\sigma(m),j} + (1-b) \sum_{i=\sigma(m)+1}^{m} A_{i,\tau(n)} - bA_{\sigma(m),\tau(n)} \\
\ge& \left\{ \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \right. \\
&+ (a-b) \sum_{j=\tau(n)+1}^{n} \left(\frac{b}{a}\right)^{j-\tau(n)} \\
&+ (1+a-b) \sum_{i=\sigma(m)+1}^{m} \sum_{j=\tau(n)+1}^{n} \frac{b^{i-\sigma(m)}}{C_{i-\sigma(m)+j-\tau(n)}^{j-\tau(n)}} \left(\frac{b}{a}\right)^{j-\tau(n)} \\
&\left. + (1-b) \sum_{i=\sigma(m)+1}^{m} b^{i-\sigma(m)} - b \right\} A_{\sigma(m),\tau(n)},
\end{aligned} \tag{2.491}$$

which leads to the required contradiction. The proof is completed. □

Noting that if $\sigma(m) = m - \sigma$ and $\tau(n) = n - \tau$, then $d_{m,n} (= d)$ is a constant. Thus, from Theorem 2.97, we can obtain the following corollary.

Corollary 2.98. *Assume that (H) holds, $\sigma(m) = m - \sigma$, and $\tau(n) = n - \tau$, where σ and τ are two positive integers. If*

$$\limsup_{m,n\to\infty} \sum_{i=m-\sigma}^{m} \sum_{j=n-\tau}^{n} p_{i,j} \frac{1}{b^{m-i}} \left(\frac{a}{b}\right)^{n-j} C_{m-i+n-j}^{n-j} > d, \tag{2.492}$$

then every solution of (2.452) *oscillates.*

In view of (H), $a \ge b$, and $b \le 1$, we have

$$\frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} \ge 1 \quad \text{for } i \le m,\ j \le n. \tag{2.493}$$

Hence, from Theorem 2.97, it is easy to obtain the next corollary.

Corollary 2.99. *Assume that (H) holds, $a \geq b$, and $b \leq 1$. If*

$$\limsup_{m,n\to\infty} \sum_{i=\sigma(m)}^{m} \sum_{j=\tau(n)}^{n} p_{i,j} > b, \tag{2.494}$$

then every solution of (2.452) *oscillates.*

If (2.476) and (2.480) do not hold, then we have the following results.

Theorem 2.100. *Assume that (H) holds, $a \geq b$, and $b \leq 1$. If*

$$\limsup_{m,n\to\infty} \frac{1}{b^{m-\sigma(m)}} \left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} = \infty, \tag{2.495}$$

$$\liminf_{m,n\to\infty} \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} \frac{1}{b^{\sigma(m)-\sigma(i)}} \left(\frac{a}{b}\right)^{\tau(n)-\tau(j)} C_{\sigma(m)-\sigma(i)+\tau(n)-\tau(j)}^{\tau(n)-\tau(j)} > 0, \tag{2.496}$$

then every solution of (2.452) *oscillates.*

Proof. In view of (2.495) and Corollary 2.91, we obtain

$$\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} \geq \frac{1}{b^{m-\sigma(m)}} \left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} \longrightarrow \infty \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty. \tag{2.497}$$

On the other hand, from (2.496) and Lemma 2.93, we have that

$$\limsup_{m,n\to\infty} \frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} \tag{2.498}$$

exists, which is a contradiction. The proof is completed. □

Corollary 2.101. *Assume that (H) holds, $a \geq b$, and $b \leq 1$. If either* $\lim_{m\to\infty}(m - \sigma(m)) = \infty$ *or* $\lim_{n\to\infty}(n - \tau(n)) = \infty$ *holds, and*

$$\liminf_{m,n\to\infty} \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} > 0, \tag{2.499}$$

then every solution of (2.452) *oscillates.*

Noting that if $a \geq b$ and $b \leq 1$ and either $\lim_{m\to\infty}(m-\sigma(m)) = \infty$ or $\lim_{n\to\infty}(n-\tau(n)) = \infty$, then (2.495) holds, and it is easy to see that (2.496) holds. Therefore, Corollary 2.101 holds.

As a matter of convenience, let

$$\lambda = \lambda_{m,n} = \frac{2(m-\sigma(m))(n-\tau(n))}{m-\sigma(m)+n-\tau(n)}, \tag{2.500}$$

$$\theta = \theta_{m,n} = \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} \frac{1}{b^{i-\sigma(i)}} \left(\frac{a}{b}\right)^{j-\tau(j)} C_{i-\sigma(i)+j-\tau(j)}^{j-\tau(j)}, \tag{2.501}$$

$$k = k_{m,n} = \frac{(m-\sigma(m))^2}{m-\sigma(m)+n-\tau(n)}, \tag{2.502}$$

$$l = l_{m,n} = \frac{(n-\tau(n))^2}{m-\sigma(m)+n-\tau(n)}, \tag{2.503}$$

$$s = s_{m,n} = \frac{2^{\lambda_{m,n}}(1+\lambda_{m,n})^{1+\lambda_{m,n}}}{\lambda_{m,n}^{\lambda_{m,n}}(m-\sigma(m))(n-\tau(n))C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)}}. \tag{2.504}$$

Theorem 2.102. *Assume that (H) holds. If* $\liminf_{m,n\to\infty}(b_{m,n}/p_{m,n}) < \infty$, *and*

$$\liminf_{m,n\to\infty} \frac{1}{b^{m-\sigma(m)}} \left(\frac{a}{b}\right)^{n-\tau(n)} C_{m-\sigma(m)+n-\tau(n)}^{n-\tau(n)} > 0, \tag{2.505}$$

$$\liminf_{m,n\to\infty} s_{m,n}\theta_{m,n} \frac{1}{b^{1+k_{m,n}}} \left(\frac{a}{b}\right)^{l_{m,n}} = \liminf_{m,n\to\infty} s\theta \frac{1}{b^{1+k}} \left(\frac{a}{b}\right)^{l} > 1, \tag{2.506}$$

then every solution of (2.452) *oscillates.*

Proof. Suppose to the contrary, there exists an eventually positive solution $\{A_{m,n}\}$ of (2.452) such that $A_{m,n} > 0$ and $p_{m,n} \geq 0$ for $m \geq \sigma^3(M)$ and $n \geq \tau^3(N)$, where M and N are two positive integers. From (2.452), we have for $m \geq M$ and $n \geq N$,

$$\frac{2\sqrt{a}}{b} \cdot \frac{(A_{m+1,n}A_{m,n+1})^{1/2}}{A_{m,n}} \leq \frac{A_{m+1,n} + aA_{m,n+1}}{bA_{m,n}} \leq 1 - p_{m,n} \frac{A_{\sigma(m),\tau(n)}}{bA_{m,n}}. \tag{2.507}$$

Hence, by means of Corollary 2.91 and the well-known inequality between the arithmetic and geometric means, we obtain

$$\begin{aligned}
&\left(\frac{2\sqrt{a}}{b}\right)^{(m-\sigma(m))(n-\tau(n))}\prod_{i=\sigma(m)}^{m-1}\prod_{j=\tau(n)}^{n-1}\frac{\left(A_{i+1,j}A_{i,j+1}\right)^{1/2}}{A_{i,j}}\\
&\quad\le\prod_{i=\sigma(m)}^{m-1}\prod_{j=\tau(n)}^{n-1}\left(1-p_{i,j}\frac{A_{\sigma(i),\tau(j)}}{bA_{i,j}}\right)\\
&\quad\le\left(1-\frac{1}{b(m-\sigma(m))(n-\tau(n))}\sum_{i=\sigma(m)}^{m-1}\sum_{j=\tau(n)}^{n-1}p_{i,j}\frac{A_{\sigma(i),\tau(j)}}{A_{i,j}}\right)^{(m-\sigma(m))(n-\tau(n))}\\
&\quad\le\left(1-\frac{1}{b(m-\sigma(m))(n-\tau(n))}\right.\\
&\qquad\left.\times\sum_{i=\sigma(m)}^{m-1}\sum_{j=\tau(n)}^{n-1}p_{i,j}\frac{1}{b^{i-\sigma(i)}}\left(\frac{a}{b}\right)^{j-\tau(j)}C_{i-\sigma(i)+j-\tau(j)}^{j-\tau(j)}\right)^{(m-\sigma(m))(n-\tau(n))}\\
&\quad=\left(1-\frac{\theta_{m,n}}{b(m-\sigma(m))(n-\tau(n))}\right)^{(m-\sigma(m))(n-\tau(n))}.
\end{aligned}\tag{2.508}$$

Since

$$\begin{aligned}
&\prod_{i=\sigma(m)}^{m-1}\prod_{j=\tau(n)}^{n-1}\frac{\left(A_{i+1,j}A_{i,j+1}\right)^{1/2}}{A_{i,j}}\\
&\quad=\prod_{i=\sigma(m)}^{m-1}\left(\frac{A_{i,n}}{A_{i,\tau(n)}}\right)^{1/2}\prod_{j=\tau(n)}^{n-1}\left(\frac{A_{m,j}}{A_{\sigma(m),j}}\right)^{1/2}\\
&\quad\ge\left(\frac{A_{m,n}}{A_{\sigma(m),\tau(n)}}\right)^{(m-\sigma(m)+n-\tau(n))/2}\prod_{i=\sigma(m)}^{m-1}\frac{1}{b^{m-i}}\cdot\frac{1}{b^{i-\sigma(m)}}\\
&\qquad\times\prod_{j=\tau(n)}^{n-1}\left(\frac{a}{b}\right)^{n-j}\cdot\left(\frac{a}{b}\right)^{j-\tau(n)}\\
&\quad=\frac{a^{(n-\tau(n))^2}}{b^{(m-\sigma(m))^2+(n-\tau(n))^2}}\left(\frac{A_{m,n}}{A_{\sigma(m),\tau(n)}}\right)^{(m-\sigma(m)+n-\tau(n))/2},
\end{aligned}\tag{2.509}$$

then from (2.508) and the inequality $x(1-x)^{\lambda} \le \lambda^{\lambda}/(1+\lambda)^{1+\lambda}$ for $x \in (0,1)$, we have

$$
\begin{aligned}
\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} &\ge \left(\frac{2\sqrt{a}}{b}\right)^{\lambda} \frac{1}{b^{2k}} \left(\frac{a}{b}\right)^{2l} \left(1 - \frac{\theta}{[b(m-\sigma(m))(n-\tau(n))]}\right)^{-\lambda} \\
&\ge \left(\frac{2\sqrt{a}}{b}\right)^{\lambda} \frac{1}{b^{2k}} \left(\frac{a}{b}\right)^{2l} \frac{\theta}{b(m-\sigma(m))(n-\tau(n))} \frac{(1+\lambda)^{1+\lambda}}{\lambda^{\lambda}} \\
&= s\theta \frac{1}{b^{k+1}} \left(\frac{a}{b}\right)^{l} \frac{1}{b^{m-\sigma(m)}} \left(\frac{a}{b}\right)^{n-\tau(n)} C^{n-\tau(n)}_{m-\sigma(m)+n-\tau(n)},
\end{aligned} \tag{2.510}
$$

where λ, θ, l, k, and s are defined by (2.500)–(2.504). In view of (2.506), there is a constant $r > 1$ such that

$$
s\theta \frac{1}{b^{k+1}} \left(\frac{a}{b}\right)^{l} > r \quad \forall \text{ large } m, n. \tag{2.511}
$$

Hence from (2.510), we have

$$
A_{\sigma(m),\tau(n)} \ge \frac{r}{b^{m-\sigma(m)}} \left(\frac{a}{b}\right)^{n-\tau(n)} C^{n-\tau(n)}_{m-\sigma(m)+n-\tau(n)} A_{m,n}. \tag{2.512}
$$

Substituting (2.512) into (2.452), we get for all large m and n,

$$
\begin{aligned}
\frac{2\sqrt{a}}{b} \cdot \frac{(A_{m+1,n}A_{m,n+1})^{1/2}}{A_{m,n}} &\le \frac{A_{m+1,n} + aA_{m,n+1}}{bA_{m,n}} \le 1 - p_{m,n} \frac{A_{\sigma(m),\tau(n)}}{bA_{m,n}} \\
&\le 1 - \frac{r}{b} p_{m,n} \frac{1}{b^{m-\sigma(m)}} \left(\frac{a}{b}\right)^{n-\tau(n)} C^{n-\tau(n)}_{m-\sigma(m)+n-\tau(n)}.
\end{aligned} \tag{2.513}
$$

Hence, for all large m and n,

$$
\begin{aligned}
&\left(\frac{2\sqrt{a}}{b}\right)^{(m-\sigma(m))(n-\tau(n))} \prod_{i=\sigma(m)}^{m-1} \prod_{j=\tau(n)}^{n-1} \frac{(A_{i+1,j}A_{i,j+1})^{1/2}}{A_{i,j}} \\
&\quad \le \prod_{i=\sigma(m)}^{m-1} \prod_{j=\tau(n)}^{n-1} \left(1 - \frac{r}{b} p_{i,j} \frac{1}{b^{i-\sigma(i)}} \left(\frac{a}{b}\right)^{j-\tau(j)} C^{n-\tau(n)}_{i-\sigma(i)+j-\tau(j)}\right).
\end{aligned} \tag{2.514}
$$

Thus, as in the above proof, for all large m and n we can obtain

$$\begin{aligned}\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} &\geq \left(\frac{2\sqrt{a}}{b}\right)^{\lambda}\frac{1}{b^{2k}}\left(\frac{a}{b}\right)^{2l}\left(1-\frac{r\theta}{[b(m-\sigma(m))(n-\tau(n))]}\right)^{-\lambda}\\ &\geq rs\theta\frac{1}{b^{k+1}}\left(\frac{a}{b}\right)^{l}\frac{1}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)}C^{n-\tau(n)}_{m-\sigma(m)+n-\tau(n)}\\ &\geq \frac{r^2}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)}C^{n-\tau(n)}_{m-\sigma(m)+n-\tau(n)}.\end{aligned}\tag{2.515}$$

By induction, we get for any positive integer N,

$$\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} \geq \frac{r^N}{b^{m-\sigma(m)}}\left(\frac{a}{b}\right)^{n-\tau(n)}C^{n-\tau(n)}_{m-\sigma(m)+n-\tau(n)},\tag{2.516}$$

for all large m and n. In view of (2.505), we get

$$\lim_{m,n\to\infty}\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} = +\infty.\tag{2.517}$$

On the other hand, in view of (2.452), we have, for all large m and n,

$$\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} \leq \frac{b_{m,n}}{p_{m,n}}.\tag{2.518}$$

Since $\liminf_{m,n\to\infty}(b_{m,n}/p_{m,n}) < \infty$, then

$$\liminf_{m,n\to\infty}\frac{A_{\sigma(m),\tau(n)}}{A_{m,n}} < +\infty,\tag{2.519}$$

which is a contradiction to (2.517). The proof is completed. □

Corollary 2.103. *Assume that (H) holds, $\sigma(m) = m-\sigma$, and $\tau(n) = n-\tau$, where σ and τ are two positive integers. If $a \geq b$, $b \leq 1$, and*

$$\liminf_{m,n\to\infty}\frac{1}{\sigma\tau}\sum_{i=m-\sigma}^{m-1}\sum_{j=n-\tau}^{n-1}p_{i,j} > \frac{\lambda^{\lambda}b^{1+k+\sigma}}{2^{\lambda}(1+\lambda)^{1+\lambda}}\cdot\left(\frac{a}{b}\right)^{l+\tau},\tag{2.520}$$

then every solution of (2.452) *is oscillatory, where $\lambda = 2\sigma\tau/(\sigma+\tau)$, and k, l are defined by* (2.502) *and* (2.503).

Example 2.104. Consider the partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{[m/2],[n/2]} = 0,\tag{2.521}$$

where $p_{m,n} = 1/(m+1)(n+1)$, $m, n = 0, 1, 2, \ldots$. We can see that if $m = 2k$, $n = 2l$, $k, l = 1, 2, \ldots$, then

$$\sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} = \sum_{i=k}^{2k-1} \sum_{j=l}^{2l-1} \frac{1}{(i+1)(j+1)} \geq \sum_{i=k}^{2k-1} \frac{1}{i+1} \cdot \frac{1}{2l} > \frac{1}{2} \cdot \frac{k}{2k} = \frac{1}{4}, \tag{2.522}$$

and in the same method, if $m = 2k$ and $n = 2l - 1$ or $m = 2k - 1$ and $n = 2l$ or $m = 2k - 1$ and $n = 2l - 1$, $k, l = 1, 2, \ldots$, then

$$\sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} \geq \frac{1}{4}. \tag{2.523}$$

Hence,

$$\liminf_{m,n\to\infty} \sum_{i=\sigma(m)}^{m-1} \sum_{j=\tau(n)}^{n-1} p_{i,j} \geq \frac{1}{4} > 0. \tag{2.524}$$

Thus, by means of Corollary 2.101, every solution of (2.521) oscillates.

2.9. Linear PDEs with positive and negative coefficients

In this section, we consider the delay partial difference equations with positive and negative coefficients of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-k,n-l} - q_{m,n}A_{m-k',n-l'} = 0, \tag{2.525}$$

where $m, n \in N_0$, and $k, k', l', l \in N_0$, $p_{m,n}, q_{m,n} \in [N_0^2, (0, \infty)]$, $k \geq k'+1$, $l \geq l'+1$.

The following lemma is a special case of Lemma 2.88.

Lemma 2.105.

$$\begin{aligned} &\sum_{i=m-k}^{m} \sum_{j=n-l}^{n} \left(A_{i+1,j} + A_{i,j+1} - A_{i,j}\right) \\ &\quad = \sum_{i=m+1-k}^{m+1} \sum_{j=n+1-l}^{n} A_{i,j} + \sum_{i=m-k}^{m} A_{i,n+1} - A_{m-k,n-l} + A_{m+1,n-l}. \end{aligned} \tag{2.526}$$

Assume that there exist positive integers s, t such that $s \geq m$, $t \geq n$, and

$$\begin{aligned} C_{s,t} &= A_{s,t} - (3)^{s+t-m-n}\left(\sum_{i=s}^{m+k'} q_{i,n}A_{i-k',n-l'} + \sum_{j=t}^{n+l'} q_{m,j}A_{m-k',j-l'}\right) \\ &\quad - \frac{1}{2}\left(\sum_{i=s}^{m+k'} q_{i+k'-k,n+l'-l}A_{i-k,n-l} + \sum_{j=t}^{n+l'} q_{m+k'-k,j+l'-l}A_{m-k,j-l}\right). \end{aligned} \tag{2.527}$$

Let

$$\alpha_{m,n} = p_{m,n} - q_{m+k'-k,n+l'-l} > 0 \quad \text{for } m \geq k - k',\ n \geq l - l'. \tag{2.528}$$

From (2.527), we obtain the following results.

Lemma 2.106. *Assume that* $\{A_{m,n}\}$ *is an eventually positive solution of* (2.525), *that is, there exist positive integers* M, N *such that* $A_{m,n} > 0$ *as* $m \geq M$, $n \geq N$. *Then*

(i) $C_{m,n}$ *is monotone decreasing in m, n, that is,*

$$C_{m+1,n} \leq C_{m,n}, \qquad C_{m,n+1} \leq C_{m,n}; \tag{2.529}$$

(ii) $C_{m,n} \leq A_{m,n}$;

(iii) $C_{m+1,n} + C_{m,n+1} - C_{m,n} = -\alpha_{m,n}A_{m-k,n-l} - \beta_{m,n}(A)$,

where

$$\begin{gathered}\beta_{m,n}(A) = 3q_{m,n}A_{m-k',n-l'} + 5\theta_1 + \frac{1}{2}\theta_2,\\ \theta_1 = \sum_{i=m+1}^{m+k'} q_{i,n}A_{i-k',n-l'} + \sum_{j=n+1}^{n+l'} q_{m,j}A_{m-k',j-l'},\\ \theta_2 = \sum_{i=m}^{m+k'} q_{i+k'-k,n+l'-l}A_{i-k,n-l} + \sum_{j=n}^{n+l'} q_{m+k'-k,j+l'-l}A_{m-k,j-l}.\end{gathered} \tag{2.530}$$

Proof. (i) From (2.527), we obtain

$$\begin{gathered}C_{m+1,n} = A_{m+1,n} - 3\theta_1 - \frac{1}{2}\theta_2 - 3q_{m,n}A_{m-k',n-l'} + \frac{1}{2}q_{m+k'-k,n+l'-l}A_{m-k,n-l},\\ C_{m,n} = A_{m,n} - \theta_1 - \frac{1}{2}\theta_2 - 2q_{m,n}A_{m-k',n-l'}.\end{gathered} \tag{2.531}$$

We note that $A_{m,n} > 0$, thus we have

$$\begin{aligned}C_{m+1,n} - C_{m,n} &\leq A_{m+1,n} + A_{m,n+1} - A_{m,n} - 2\theta_1 - q_{m,n}A_{m-k',n-l'}\\ &\quad + \frac{1}{2}q_{m+k'-k,n+l'-l}A_{m-k,n-l}\\ &< -p_{m,n}A_{m-k,n-l} + q_{m,n}A_{m-k',n-l'} - 2\theta_1\\ &\quad - q_{m,n}A_{m-k',n-l'} + q_{m+k'-k,n+l'-l}A_{m-k,n-l}\\ &= -\alpha_{m,n}A_{m-k,n-l} - 2\theta_1 \leq -\alpha_{m,n}A_{m-k,n-l} \leq 0,\end{aligned} \tag{2.532}$$

that is, $C_{m+1,n} - C_{m,n} < 0$. Similarly, we have also $C_{m,n+1} - C_{m,n} < 0$.

(ii) From (2.531), we immediately obtain (ii).
(iii) From (2.527), we have

$$C_{m,n+1} = A_{m,n+1} - 3\theta_1 - \frac{1}{2}\theta_2 - 3q_{m,n}A_{m-k',n-l'} + \frac{1}{2}q_{m+k'-k,n+l'-l}A_{m-k,n-l}. \tag{2.533}$$

By the above equality and (2.531), we obtain

$$\begin{aligned} C_{m+1,n} + C_{m,n+1} - C_{m,n} &= -\alpha_{m,n}A_{m-k,n-l} - 3q_{m,n}A_{m-k',n-l'} - 5\theta_1 - \frac{1}{2}\theta_2 \\ &= -\alpha_{m,n}A_{m-k,n-l} - \beta_{m,n}(A). \end{aligned} \tag{2.534}$$

Hence, $C_{m+1,n} + C_{m,n+1} - C_{m,n} = -\alpha_{m,n}A_{m-k,n-l} - \beta_{m,n}(A)$. Note that $\beta_{m,n}(A) > 0$, thus we also have

$$C_{m+1,n} + C_{m,n+1} - C_{m,n} < -\alpha_{m,n}A_{m-k,n-l} < 0. \tag{2.535}$$

□

Lemma 2.107. *Assume that* (2.528) *holds. Further, assume that for* $m \geq k - k'$, $n \geq l - l'$,

$$\left(\sum_{i=m}^{m+k'} q_{i,n} + \sum_{j=n}^{n+l'} q_{m,j}\right) + \frac{1}{2}\left(\sum_{i=m}^{m+k'} q_{i+k'-k,n+l'-l} + \sum_{j=n}^{n+l'} q_{m+k'-k,j+l'-l}\right) < 1. \tag{2.536}$$

Let $\{A_{m,n}\}$ *be an eventually positive solution of* (2.525). *Then* $\{C_{m,n}\}$ *defined by* (2.527) *is decreasing and eventually positive in* m, n.

Proof. By Lemma 2.106, $\{C_{m,n}\}$ is decreasing in m, n. Next, we will show that $\{C_{m,n}\}$ is eventually positive in m, n. Because $\{C_{m,n}\}$ is monotone decreasing in m, n, thus the limit $\lim_{m,n\to\infty} C_{m,n}$ exists. If $\lim_{m,n\to\infty} C_{m,n} = -\infty$, then $\{A_{m,n}\}$ must be unbounded. Hence, there exists a double sequence $\{(m_k, n_k)\}$ such that $\lim_{k\to\infty} m_k = \infty$, $\lim_{k\to\infty} n_k = \infty$, $A_{m_k,n_k} = \max_{N\leq n\leq n_k+l, M\leq m\leq m_k+k}\{A_{m-k,n-l}\}$ and

$$\lim_{k\to\infty} A_{m_k,m_k} = \infty. \tag{2.537}$$

On the other hand, we have

$$\begin{aligned}
C_{m_k,n_k} &= A_{m_k,n_k} - \left(\sum_{i=m_k}^{m_k+k'} q_{i,n_k}A_{i-k',n_k-l'} + \sum_{j=n_k}^{n_k+l'} q_{m_k,j}A_{m_k-k',j-l'}\right) \\
&\quad - \frac{1}{2}\left(\sum_{i=m_k}^{m_k+k'} q_{i+k'-k,n_k+l'-l}A_{i-k,n_k-l} + \sum_{j=n_k}^{n_k+l'} q_{m_k+k'-k,j+l'-l}A_{m_k-k,j-l}\right) \\
&\geq A_{m_k,n_k}\left[1 - \left(\sum_{i=m_k}^{m_k+k'} q_{i,n_k} + \sum_{j=n_k}^{n_k+l'} q_{m_k,j}\right)\right. \\
&\quad \left. - \frac{1}{2}\left(\sum_{i=m_k}^{m_k+k'} q_{i+k'-k,n_k+l'-l} + \sum_{j=n_k}^{n_k+l'} q_{m_k+k'-k,j+l'-l}\right)\right] \geq 0,
\end{aligned} \tag{2.538}$$

a contradiction. Hence $\lim_{m,n\to\infty} C_{m,n} = \beta$ exists, where β is finite. As before, if $\{A_{m,n}\}$ is unbounded, then $\beta \geq 0$. Now we consider the case that $\{A_{m,n}\}$ is bounded. Let $\bar{\beta} = \limsup_{m,n\to\infty} A_{m,n} = \lim_{m',n'\to\infty} A_{m',n'}$. Then

$$\begin{aligned}
A_{m',n'} - C_{m',n'} &= \left(\sum_{i=m'}^{m'+k'} q_{i,n'}A_{i-k',n'-l'} + \sum_{j=n'}^{n'+l'} q_{m',j}A_{m'-k',j-l'}\right) \\
&\quad + \frac{1}{2}\left(\sum_{i=m'}^{m'+k'} q_{i+k'-k,n'+l'-l}A_{i-k,n'-l} + \sum_{j=n'}^{n'+l'} q_{m'+k'-k,j+l'-l}A_{m'-k,j-l}\right) \\
&\leq A(\xi_m,\eta_n)\left[\left(\sum_{i=m'}^{m'+k'} q_{i,n'} + \sum_{j=n'}^{n'+l'} q_{m',j}\right)\right. \\
&\quad \left. + \frac{1}{2}\left(\sum_{i=m'}^{m'+k'} q_{i+k'-k,n'+l'-l} + \sum_{j=n'}^{n'+l'} q_{m'+k'-k,j+l'-l}\right)\right] \\
&\leq A(\xi_m,\eta_n),
\end{aligned} \tag{2.539}$$

where $A(\xi'_m,\eta'_n) = \max\{A_{i-k,j-l} \mid i = m', m'+1,\ldots,m'+k',\ j = n', n'+1,\ldots,n'+l'\}$. Taking superior limit on both sides of the above inequality, we have $\bar{\beta} - \beta \leq \bar{\beta}$, therefore $\beta \geq 0$. Hence $C_{m,n} > 0$ for $m \geq M$, $n \geq N$. □

Theorem 2.108. *Assume that* (2.528) *and* (2.536) *hold. Further, assume that either*

$$\liminf_{m,n\to\infty}\left(\frac{1}{kl}\sum_{i=m-k}^{m-1}\sum_{j=n-l}^{n-1}\left(p_{i,j} - q_{i-k+k',j-l+l'}\right)\right) > \frac{\omega^{\omega}}{(\omega+1)^{\omega+1}}, \tag{2.540}$$

where $\omega = \max(k, l)$, *or for all large m and n*

$$\left(p_{m,n} - q_{m-k+k',n-l+l'}\right) \geq \xi > \frac{(k+l)^{k+l}}{(k+l+1)^{k+l+1}}. \tag{2.541}$$

Then every solution of (2.525) *oscillates.*

Proof. Suppose to the contrary, assume that (2.525) has an eventually positive solution $\{A_{m,n}\}$. By Lemmas 2.106 and 2.107, it follows that the sequence $\{C_{m,n}\}$ is eventually decreasing and positive and

$$C_{m+1,n} + C_{m,n+1} - C_{m,n} + \left(p_{m,n} - q_{m-k+k',n-l+l'}\right)A_{m-k,n-l} \leq 0. \tag{2.542}$$

Hence, we have

$$C_{m+1,n} + C_{m,n+1} - C_{m,n} + \left(p_{m,n} - q_{m-k+k',n-l+l'}\right)C_{m-k,n-l} \leq 0. \tag{2.543}$$

In view of (2.540) and (2.541), by Corollaries 2.18 and 2.60, difference inequality (2.543) cannot have an eventually positive solution. The proof is complete. □

Example 2.109. Consider the partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \left(\frac{3}{4} - \frac{1}{2n}\right)A_{m-2,n-1} - \frac{1}{n}A_{m-1,n} = 0, \tag{2.544}$$

where $m \geq 2$, $n \geq 4$, $p_{m,n} = 3/4 - 1/2n$, $q_{m,n} = 1/n$, $k = 2$, $k' = l = 1$, $l' = 0$. Since $k = 2 > 1 = k'$, $l > l'$ and for $m \geq 2$, $n \geq 4$, we have

$$\begin{aligned}
&p_{m,n} - q_{m-k+k',n-l+l'} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{n-1} > 0,\\
&\liminf_{m,n\to\infty}\left[\frac{1}{kl}\sum_{i=m-k}^{m-1}\sum_{j=n-l}^{n-1}\left(p_{i,j} - q_{i-k+k',j-l+l'}\right)\right]\\
&\quad = \liminf_{m,n\to\infty}\left[\frac{1}{2}\sum_{i=m-2}^{m-1}\sum_{j=n-1}^{n-1}\left(\frac{3}{4} - \frac{1}{2j} - \frac{1}{j-1}\right)\right]\\
&\quad = \liminf_{m,n\to\infty}\left(\frac{3}{4} - \frac{1}{2(n-1)} - \frac{1}{n-2}\right) = \frac{3}{4} > \frac{4}{27} = \frac{\omega^{\omega}}{(\omega+1)^{\omega+1}}.
\end{aligned} \tag{2.545}$$

Hence, all the hypotheses of Theorem 2.108 are satisfied. Therefore, all solutions of (2.544) are oscillatory. In fact, (2.544) has an oscillatory solution $\{A_{mn}\} = \{(-1)^m(1/2^n)\}$ for $m \geq 2$, $n \geq 4$.

2.10. Nonexistence of monotone solutions of neutral PDEs

We consider the partial difference equation of the form

$$T(\Delta_m, \Delta_n)(A_{m,n} - p_{m,n}A_{m-r,n-h}) + q_{m,n}A_{m-k,n-l} = 0, \tag{2.546}$$

where $T(\Delta_m, \Delta_n) = a\Delta_m\Delta_n + b\Delta_m + c\Delta_n + dI$, a, b, c, d are nonnegative constants, $\Delta_m A_{m,n} = A_{m+1,n} - A_{m,n}$, $\Delta_n A_{m,n} = A_{m,n+1} - A_{m,n}$, and $IA_{m,n} = A_{m,n}$. The delays r, h, k, l are positive integers, $0 \le p_{m,n} \le 1$ and $q_{m,n} \ge 0$ on N_0^2.

By a solution of (2.546), we mean a nontrivial double sequence $\{A_{m,n}\}$ satisfying (2.546) for $m \ge m_0$, $n \ge n_0$. A sequence $\{A_{m,n}\}$ is nondecreasing (nonincreasing) if $\Delta_m A_{m,n} \ge (\le) 0$ and $\Delta_n A_{m,n} \ge (\le) 0$. A solution $\{A_{m,n}\}$ is called to be a monotone solution, if it is either nondecreasing or nonincreasing.

Throughout this section, we assume that
 (i) $a \ge 0$, $d \ge 0$, $b, c > a$, $b + c > a + d$;
 (ii) $0 \le p_{m,n} \le 1$ and $q_{m,n} \ge 0$ on N_0^2 and

$$\limsup_{m,n\to\infty} q_{m,n} > 0. \tag{2.547}$$

For the sake of convenience, we set $p_1 = a$, $p_2 = b - a$, $p_3 = c - a$, $p_4 = b + c - a - d$. Furthermore, we define the set E by

$$E = \{\lambda > 0 \mid p_4 - \lambda q_{m,n} > 0 \text{ eventually}\}. \tag{2.548}$$

Theorem 2.110. *Assume that there exist integers $M \ge m_0$, $N \ge n_0$ such that one of the following conditions holds:*
 (i) *for $k > l$ and $r > h$,*

$$\begin{aligned}\inf_{\lambda\in E,\, m\ge M,\, n\ge N} \Bigg\{ &\frac{1}{\lambda}\left(p_1 + \frac{2p_2p_3}{p_4}\right)^l \left(\frac{p_2}{p_4}\right)^{k-l} \times \left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}(p_4 - \lambda q_{m-i-j,n-i})\right]^{-1/(k-l)} \\ &+ p_{m-k,n-l}\left(p_1 + \frac{2p_2p_3}{p_4}\right)^h \left(\frac{p_2}{p_4}\right)^{r-h} \\ &\times \left[\prod_{j=1}^{r-h}\prod_{i=1}^{h}(p_4 - \lambda q_{m-i-j,n-i})\right]^{-1/(r-h)} \Bigg\} > 1;\end{aligned} \tag{2.549}$$

(ii) *for $k > l$ and $r < h$,*

$$\inf_{\lambda\in E,\, m\ge M,\, n\ge N}\left\{\frac{1}{\lambda}\left(p_1+\frac{2p_2p_3}{p_4}\right)^{l}\left(\frac{p_2}{p_4}\right)^{k-l}\times\left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}\left(p_4-\lambda q_{m-i-j,n-i}\right)\right]^{-1/(k-l)}\right.$$
$$+\,p_{m-k,n-l}\left(p_1+\frac{2p_2p_3}{p_4}\right)^{r}\left(\frac{p_3}{p_4}\right)^{h-r}$$
$$\left.\times\left[\prod_{j=1}^{h-r}\prod_{i=1}^{r}\left(p_4-\lambda q_{m-i,n-i-j}\right)\right]^{-1/(h-r)}\right\}>1; \tag{2.550}$$

(iii) *for $k < l$ and $r > h$,*

$$\inf_{\lambda\in E,\, m\ge M,\, n\ge N}\left\{\frac{1}{\lambda}\left(p_1+\frac{2p_2p_3}{p_4}\right)^{k}\left(\frac{p_3}{p_4}\right)^{l-k}\times\left[\prod_{j=1}^{l-k}\prod_{i=1}^{k}\left(p_4-\lambda q_{m-i,n-i-j}\right)\right]^{-1/(l-k)}\right.$$
$$+\,p_{m-k,n-l}\left(p_1+\frac{2p_2p_3}{p_4}\right)^{h}\left(\frac{p_2}{p_4}\right)^{r-h}$$
$$\left.\times\left[\prod_{j=1}^{r-h}\prod_{i=1}^{h}\left(p_4-\lambda q_{m-i-j,n-i}\right)\right]^{-1/(r-h)}\right\}>1; \tag{2.551}$$

(iv) *for $k < l$ and $r < h$,*

$$\inf_{\lambda\in E,\, m\ge M,\, n\ge N}\left\{\frac{1}{\lambda}\left(p_1+\frac{2p_2p_3}{p_4}\right)^{k}\left(\frac{p_3}{p_4}\right)^{l-k}\times\left[\prod_{j=1}^{l-k}\prod_{i=1}^{k}\left(p_4-\lambda q_{m-i,n-i-j}\right)\right]^{-1/(l-k)}\right.$$
$$+\,p_{m-k,n-l}\left(p_1+\frac{2p_2p_3}{p_4}\right)^{r}\left(\frac{p_3}{p_4}\right)^{h-r}$$
$$\left.\times\left[\prod_{j=1}^{h-r}\prod_{i=1}^{r}\left(p_4-\lambda q_{m-i,n-i-j}\right)\right]^{-1/(h-r)}\right\}>1. \tag{2.552}$$

Then, (2.546) *has no eventually positive (negative) and nondecreasing (nonincreasing) solution.*

Proof. Let $\{A_{m,n}\}$ be an eventually positive and nondecreasing solution of (2.546). Then

$$A_{m,n}\ge A_{m-1,n}\ge\cdots\ge A_{m-r,n}\ge A_{m-r,n-1}\ge\cdots\ge A_{m-r,n-h}\ge p_{m,n}A_{m-r,n-h}. \tag{2.553}$$

Let

$$\omega_{m,n} = A_{m,n} - p_{m,n}A_{m-r,n-h}. \tag{2.554}$$

Then,

$$0 \leq \omega_{m,n} \leq A_{m,n}, \tag{2.555}$$

$$T(\Delta_m, \Delta_n)\omega_{m,n} = -q_{m,n}A_{m-k,n-l} \leq 0, \tag{2.556}$$

which implies

$$a\omega_{m+1,n+1} + (b-a)\omega_{m+1,n} + (c-a)\omega_{m,n+1} \leq (b+c-a-d)\omega_{m,n}, \tag{2.557}$$

that is,

$$p_1\omega_{m+1,n+1} + p_2\omega_{m+1,n} + p_3\omega_{m,n+1} \leq p_4\omega_{m,n}. \tag{2.558}$$

Define the set $S(\lambda)$ as follows:

$$S(\lambda) = \{\lambda > 0 \mid T(\Delta_m, \Delta_n)\omega_{m,n} + \lambda q_{m,n}\omega_{m,n} \leq 0 \text{ eventually}\}. \tag{2.559}$$

From (2.558), we have

$$\omega_{m+1,n+1} \leq \frac{p_4}{p_2}\omega_{m,n+1}, \qquad \omega_{m+1,n+1} \leq \frac{p_4}{p_3}\omega_{m+1,n}. \tag{2.560}$$

Hence, we obtain

$$\begin{aligned}\omega_{m,n} &\leq \frac{p_4}{p_2}\omega_{m-1,n} \leq \cdots \leq \left(\frac{p_4}{p_2}\right)^k \omega_{m-k,n} \\ &\leq \left(\frac{p_4}{p_2}\right)^k \left(\frac{p_4}{p_3}\right)\omega_{m-k,n-1} \leq \cdots \leq \left(\frac{p_4}{p_2}\right)^k \left(\frac{p_4}{p_3}\right)^l \omega_{m-k,n-l}.\end{aligned} \tag{2.561}$$

From (2.555) and (2.556), we have

$$T(\Delta_m, \Delta_n)\omega_{m,n} = -q_{m,n}A_{m-k,n-l} \leq -q_{m,n}\omega_{m-k,n-l} \leq -\left(\frac{p_2}{p_4}\right)^k \left(\frac{p_3}{p_4}\right)^l q_{m,n}\omega_{m,n}, \tag{2.562}$$

which implies $(p_2/p_4)^k(p_3/p_4)^l \in S(\lambda)$. Hence, $S(\lambda)$ is nonempty. For $\lambda \in S(\lambda)$, we have eventually

$$p_1\omega_{m+1,n+1} + p_2\omega_{m+1,n} + p_3\omega_{m,n+1} - (p_4 - \lambda q_{m,n})\omega_{m,n} \leq 0. \tag{2.563}$$

Hence,

$$p_4 - \lambda q_{m,n} > 0, \tag{2.564}$$

which implies that $S(\lambda) \subset E$. Due to the condition (2.547), the set E is bounded, and hence $S(\lambda)$ is bounded.

We consider the following cases.

(i) $k > l$ and $r > h$. Let $\mu \in S(\lambda)$. By (2.560), we have

$$\left(p_1 + \frac{2p_2p_3}{p_4}\right)\omega_{m+1,n+1} \le p_1\omega_{m+1,n+1} + p_2\omega_{m+1,n} + p_3\omega_{m,n+1} \le \left(p_4 - \mu q_{m,n}\right)\omega_{m,n}, \tag{2.565}$$

and hence

$$\omega_{m,n} \le \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l}\prod_{i=1}^{l}\left(p_4 - \mu q_{m-i,n-i}\right)\omega_{m-l,n-l}. \tag{2.566}$$

For $j = 1, 2, \dots, k - l$, we have

$$\begin{aligned}\omega_{m-j,n} &\le \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l}\prod_{i=1}^{l}\left(p_4 - \mu q_{m-i-j,n-i}\right)\omega_{m-l-j,n-l}\\ &\le \left[\left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l}\prod_{i=1}^{l}\left(p_4 - \mu q_{m-i-j,n-i}\right)\right]\left(\frac{p_4}{p_2}\right)^{k-l-j}\omega_{m-k,n-l}.\end{aligned} \tag{2.567}$$

Now, from (2.560) and (2.567), it follows that

$$\begin{aligned}\omega_{m,n}^{k-l} &\le \prod_{j=1}^{k-l}\left(\frac{p_4}{p_2}\right)^{j}\omega_{m-j,n}\\ &\le \prod_{j=1}^{k-l}\left\{\left(\frac{p_4}{p_2}\right)^{j}\left[\left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l}\prod_{i=1}^{l}\left(p_4 - \mu q_{m-i-j,n-i}\right)\right]\right.\\ &\qquad\left.\times\left(\frac{p_4}{p_2}\right)^{k-l-j}\omega_{m-k,n-l}\right\}\\ &= \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-l(k-l)}\left(\frac{p_4}{p_2}\right)^{(k-l)^2}\times\left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}\left(p_4 - \mu q_{m-i-j,n-i}\right)\right]\omega_{m-k,n-l}^{k-l},\end{aligned} \tag{2.568}$$

that is,

$$\omega_{m-k,n-l} \geq \left(p_1 + \frac{2p_2p_3}{p_4}\right)^l \left(\frac{p_2}{p_4}\right)^{k-l} \times \left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}\left(p_4 - \mu q_{m-i-j,n-i}\right)\right]^{-1/(k-l)} \omega_{m,n}. \tag{2.569}$$

Similarly,

$$\omega_{m-r,n-h} \geq \left(p_1 + \frac{2p_2p_3}{p_4}\right)^h \left(\frac{p_2}{p_4}\right)^{r-h} \times \left[\prod_{j=1}^{r-h}\prod_{i=1}^{h}\left(p_4 - \mu q_{m-i-j,n-i}\right)\right]^{-1/(r-h)} \omega_{m,n}. \tag{2.570}$$

From (2.549), there exists a constant $\alpha_1 > 1$ such that

$$\begin{aligned}\inf_{\lambda\in E,\, m\geq M,\, n\geq N} \Bigg\{ &\frac{1}{\lambda}\left(p_1 + \frac{2p_2p_3}{p_4}\right)^l \left(\frac{p_2}{p_4}\right)^{k-l} \times \left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}\left(p_4 - \lambda q_{m-i-j,n-i}\right)\right]^{-1/(k-l)} \\ &+ p_{m-k,n-l}\left(p_1 + \frac{2p_2p_3}{p_4}\right)^h \left(\frac{p_2}{p_4}\right)^{r-h} \\ &\times \left[\prod_{j=1}^{r-h}\prod_{i=1}^{h}\left(p_4 - \lambda q_{m-i-j,n-i}\right)\right]^{-1/(r-h)} \Bigg\} > \alpha_1.\end{aligned} \tag{2.571}$$

We will show that $\alpha_1\mu \in S(\lambda)$. In fact, $\mu \in S(\lambda)$ implies that

$$T(\Delta_m, \Delta_n)\omega_{m,n} + \mu q_{m,n}\omega_{m,n} \leq 0. \tag{2.572}$$

From (2.556), we have

$$\begin{aligned} 0 &\geq T(\Delta_m, \Delta_n)\omega_{m,n} + \mu q_{m,n}\omega_{m,n} \\ &= -q_{m,n}A_{m-k,n-l} + \mu q_{m,n}\omega_{m,n} = q_{m,n}\left(\mu\omega_{m,n} - A_{m-k,n-l}\right), \end{aligned} \tag{2.573}$$

and hence $\mu\omega_{m,n} \le A_{m-k,n-l}$. By (2.554), (2.556), (2.569), and (2.570), we see that

$$\begin{aligned}
T(\Delta_m,\Delta_n)\omega_{m,n} &= -q_{m,n}A_{m-k,n-l} = -q_{m,n}(\omega_{m-k,n-l} + p_{m-k,n-l}A_{m-k-r,n-l-h}) \\
&\le -q_{m,n}(\omega_{m-k,n-l} + p_{m-k,n-l}\mu\omega_{m-r,n-h}) \\
&\le -q_{m,n}\omega_{m,n}\Bigg\{\left(p_1 + \frac{2p_2p_3}{p_4}\right)^{l}\left(\frac{p_2}{p_4}\right)^{k-l} \\
&\qquad \times \left[\prod_{j=1}^{k-l}\prod_{i=1}^{l}(p_4 - \mu q_{m-i-j,n-i})\right]^{-1/(k-l)} \\
&\qquad + p_{m-k,n-l}\mu\left(p_1 + \frac{2p_2p_3}{p_4}\right)^{h}\left(\frac{p_2}{p_4}\right)^{r-h} \\
&\qquad \times \left[\prod_{j=1}^{r-h}\prod_{i=1}^{h}(p_4 - \mu q_{m-i-j,n-i})\right]^{-1/(r-h)}\Bigg\}.
\end{aligned} \tag{2.574}$$

Combining (2.571) and (2.574), we have

$$T(\Delta_m,\Delta_n)\omega_{m,n} \le -\alpha_1\mu q_{m,n}\omega_{m,n}, \tag{2.575}$$

that is, $\alpha_1\mu \in S(\lambda)$. Repeating the above argument with μ replaced by $\alpha_1\mu$, we obtain $\alpha_1^{\theta}\mu \in S(\lambda)$, $\theta = 1,2,\dots$, where $\alpha_1 > 1$. This contradicts the boundedness of $S(\lambda)$. The proof of (i) is complete.

(ii) $k > l$ and $r < h$. From (2.565), we have

$$\omega_{m,n} \le \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-r}\prod_{i=1}^{r}(p_4 - \mu q_{m-i,n-i})\omega_{m-r,n-r}. \tag{2.576}$$

For $j = 1,2,\dots,h-r$, we have

$$\begin{aligned}
\omega_{m,n-j} &\le \left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-r} \times \prod_{i=1}^{r}(p_4 - \mu q_{m-i,n-i-j})\omega_{m-r,n-r-j} \\
&\le \left[\left(p_1 + \frac{2p_2p_3}{p_4}\right)^{-r}\prod_{i=1}^{r}(p_4 - \mu q_{m-i,n-i-j})\right] \\
&\quad \times \left(\frac{p_4}{p_3}\right)^{h-r-j}\omega_{m-r,n-h}.
\end{aligned} \tag{2.577}$$

Now, from (2.560) and (2.577), it follows that

$$\begin{aligned}
\omega_{m,n}^{h-r} &\le \prod_{j=1}^{h-r}\left(\frac{p_4}{p_3}\right)^{j}\omega_{m,n-j}\\
&\le \prod_{j=1}^{h-r}\left\{\left(\frac{p_4}{p_3}\right)^{j}\left[\left(p_1+\frac{2p_2p_3}{p_4}\right)^{-r}\prod_{i=1}^{r}\left(p_4-\mu q_{m-i,n-i-j}\right)\right]\right.\\
&\qquad\left.\times\left(\frac{p_4}{p_3}\right)^{h-r-j}\omega_{m-r,n-h}\right\}\\
&=\left(p_1+\frac{2p_2p_3}{p_4}\right)^{-r(h-r)}\left(\frac{p_4}{p_3}\right)^{(h-r)^2}\\
&\quad\times\left[\prod_{j=1}^{h-r}\prod_{i=1}^{r}\left(p_4-\mu q_{m-i,n-i-j}\right)\right]\omega_{m-r,n-h}^{h-r},
\end{aligned}\tag{2.578}$$

that is,

$$\begin{aligned}
\omega_{m-r,n-h} &\ge \left(p_1+\frac{2p_2p_3}{p_4}\right)^{r}\left(\frac{p_3}{p_4}\right)^{h-r}\\
&\quad\times\left[\prod_{j=1}^{h-r}\prod_{i=1}^{r}\left(p_4-\mu q_{m-i,n-i-j}\right)\right]^{-1/(h-r)}\omega_{m,n}.
\end{aligned}\tag{2.579}$$

The rest of the proof is similar to that of (i), and thus is omitted.

(iii) $k<l$ and $r>h$. We only need to note that (2.569) now changes to

$$\begin{aligned}
\omega_{m-k,n-l} &\ge \left(p_1+\frac{2p_2p_3}{p_4}\right)^{k}\left(\frac{p_3}{p_4}\right)^{l-k}\\
&\quad\times\left[\prod_{j=1}^{l-k}\prod_{i=1}^{k}\left(p_4-\mu q_{m-i,n-i-j}\right)\right]^{-1/(l-k)}\omega_{m,n}.
\end{aligned}\tag{2.580}$$

The rest of the proof is similar to that of (i), and thus is omitted.

(iv) $k<l$ and $r<h$. We only need to note that (2.569) and (2.570) now change to (2.580) and (2.579), respectively. The rest of the proof is similar to that of (i), and thus is omitted. The proof of Theorem 2.110 is complete. □

From Theorem 2.110, we can derive some explicit sufficient conditions for the nonexistence of monotone solutions of (2.546).

Corollary 2.111. *Assume that* $k > l$, $r > h$, $p_{m,n} \geq p_0$, *and*

$$\left(p_1 + \frac{2p_2p_3}{p_4}\right)^l \left(\frac{p_2}{p_4}\right)^{k-l} \frac{\bar{q}(l+1)^{l+1}}{p_4^{l+1}l^l} + p_0\left(p_1 + \frac{2p_2p_3}{p_4}\right)^h \left(\frac{p_2}{p_4}\right)^{r-h} \frac{\bar{q}(h+1)^{h+1}}{p_4^{h+1}h^h} > 1, \tag{2.581}$$

where

$$\liminf_{m,n\to\infty} q_{m,n} = \bar{q}. \tag{2.582}$$

Then the conclusion of Theorem 2.110 holds.

2.11. Existence of positive solutions of linear PDEs

2.11.1. Equation with delay type

Consider the linear partial difference equation

$$aA_{m+1,n+1} + bA_{m,n+1} + cA_{m+1,n} - dA_{m,n} + P_{m,n}A_{m-k,n-l} = 0, \tag{2.583}$$

where $P_{m,n} > 0$ on N_0^2, $k, l \in N_0$. Throughout this paper, we assume that a, b, c, d are positive constants. The oscillation of (2.583) has been studied in Section 2.5. In the following, we mainly consider the existence of positive solutions of (2.583).

Theorem 2.112. *Assume that* $a \geq d$, $b \geq d$, $c \geq d$, *and one of the following three conditions holds:*

(i) *there exists a positive double sequence* $\{\lambda_{m,n}\}$ *such that for all sufficiently large* m, n,

$$\begin{aligned}\frac{1}{d\lambda_{m,n}}\Bigg\{&\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(a-d)\lambda_{m+1+i,n+1+i+j} + \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} b\lambda_{m+i,n+1+i+j}\\ &+ \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c\lambda_{m+1+i,n+i+j} + \sum_{j=0}^{\infty}(b-d)\lambda_{m,n+1+j}\\ &+ \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} P_{m+i,n+i+j}\lambda_{m-k+i,n-l+i+j}\Bigg\} \leq 1;\end{aligned} \tag{2.584}$$

(ii) *there exists a positive double sequence* $\{\lambda_{m,n}\}$ *such that for all sufficiently large m, n,*

$$\begin{aligned}\frac{1}{d\lambda_{m,n}}\Bigg\{&\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(a-d)\lambda_{m+1+i+j,n+1+i}+\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}b\lambda_{m+i+j,n+1+i}\\&+\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}c\lambda_{m+1+i+j,n+i}+\sum_{j=0}^{\infty}(c-d)\lambda_{m+1+j,n}\\&+\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}P_{m+i+j,n+i}\lambda_{m-k+i+j,n-l+i}\Bigg\}\leq 1;\end{aligned}\tag{2.585}$$

(iii) *there exists a positive double sequence* $\{\lambda_{m,n}\}$ *such that for all sufficiently large m, n,*

$$\begin{aligned}\frac{1}{d\lambda_{m,n}}\Bigg\{&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\lambda_{m+1+i,n+1+j}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(b-d)\lambda_{m+i,n+1+j}\\&+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}c\lambda_{m+1+i,n+j}+\sum_{i=0}^{\infty}(c-d)\lambda_{m+1+i,n}\\&+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_{m+i,n+j}\lambda_{m-k+i,n-l+j}\Bigg\}\leq 1.\end{aligned}\tag{2.586}$$

Then (2.583) *has an eventually positive solution* $\{A_{m,n}\}$ *which satisfies* $0 < A_{m,n} \leq \lambda_{m,n}$.

Proof. We only give the proof of (i), and the other cases are similar.

Let X be the set of all real bounded double sequence $y = \{y_{m,n}\}_{m=m_0,\,n=n_0}^{\infty,\infty}$ with the norm $\|y\| = \sup_{m\geq m_0,\,n\geq n_0}|y_{m,n}| < \infty$. X is a Banach space. We define a subset Ω of X as follows:

$$\Omega = \{y = \{y_{m,n}\} \in X \mid 0 \leq y_{m,n} \leq 1,\ m \geq m_0,\ n \geq n_0\}\tag{2.587}$$

and define a partial order on X in the usual way, that is,

$$x, y \in X, \quad x \leq y \text{ means that } x_{m,n} \leq y_{m,n} \text{ for } m \geq m_0,\ n \geq n_0.\tag{2.588}$$

It is easy to see that for any subset S of Ω, there exist $\inf S$ and $\sup S$. We choose $m_1 > m_0$, $n_1 > n_0$ sufficiently large such that (i) holds.

Set

$$\begin{aligned} D &= N_{m_0} \times N_{n_0}, & D_1 &= N_{m_1} \times N_{n_1}, \\ D_2 &= (N_{m_0} \times N_{n_1}) \setminus D_1, & D_3 &= (N_{m_1} \times N_{n_0}) \setminus D_1, \\ D_4 &= D \setminus (D_1 \cup D_2 \cup D_3). \end{aligned} \tag{2.589}$$

Clearly, $D = D_1 \cup D_2 \cup D_3 \cup D_4$. Define a mapping $T : \Omega \to X$ as follows:

$$Ty_{m,n} = \begin{cases} \dfrac{1}{d\lambda_{m,n}} \\ \quad \times \Bigg\{ \displaystyle\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (a-d)\lambda_{m+1+i,n+1+i+j}\, y_{m+1+i,n+1+i+j} \\ \qquad + \displaystyle\sum_{j=0}^{\infty} \sum_{i=1}^{\infty} b\lambda_{m+i,n+1+i+j}\, y_{m+i,n+1+i+j} \\ \qquad + \displaystyle\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c\lambda_{m+1+i,n+i+j}\, y_{m+1+i,n+i+j} \\ \qquad + \displaystyle\sum_{j=0}^{\infty} (b-d)\lambda_{m,n+1+j}\, y_{m,n+1+j} \\ \qquad + \displaystyle\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} P_{m+i,n+i+j}\lambda_{m-k+i,n-l+i+j} \\ \qquad \times y_{m-k+i,n-l+i+j} \Bigg\}, & (m,n) \in D_1, \\ \dfrac{n}{n_1} Ty_{m_1,n} + \left(1 - \dfrac{n}{n_1}\right), & (m,n) \in D_2, \\ \dfrac{m}{m_1} Ty_{m,n_1} + \left(1 - \dfrac{m}{m_1}\right), & (m,n) \in D_3, \\ \dfrac{mn}{m_1 n_1} Ty_{m_1,n_1} + \left(1 - \dfrac{mn}{m_1 n_1}\right), & (m,n) \in D_4. \end{cases} \tag{2.590}$$

From (2.590) and noting that $y_{m,n} \le 1$, we have

$$0 \le Ty_{m,n} \le \frac{1}{d\lambda_{m,n}} \Bigg\{ \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(a-d)\lambda_{m+1+i,n+1+i+j} + \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} b\lambda_{m+i,n+1+i+j} + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c\lambda_{m+1+i,n+i+j} + \sum_{j=0}^{\infty}(b-d)\lambda_{m,n+1+j} + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} P_{m+i,n+i+j}\lambda_{m-k+i,n-l+i+j} \Bigg\} \le 1, \quad \text{for } (m,n) \in D_1 \tag{2.591}$$

and $0 \le Ty_{m,n} \le 1$ for $(m,n) \in D_2 \cup D_3 \cup D_4$. Therefore, $T\Omega \subset \Omega$. Clearly, T is nondecreasing. By Theorem 1.9, there is a $y \in \Omega$ such that $Ty = y$, that is,

$$y_{m,n} = \begin{cases} \dfrac{1}{d\lambda_{m,n}} \times \Bigg\{ \displaystyle\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(a-d)\lambda_{m+1+i,n+1+i+j}\, y_{m+1+i,n+1+i+j} + \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} b\lambda_{m+i,n+1+i+j}\, y_{m+i,n+1+i+j} + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c\lambda_{m+1+i,n+i+j}\, y_{m+1+i,n+i+j} + \sum_{j=0}^{\infty}(b-d)\lambda_{m,n+1+j}\, y_{m,n+1+j} + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} P_{m+i,n+i+j}\lambda_{m-k+i,n-l+i+j} \times y_{m-k+i,n-l+i+j} \Bigg\}, & (m,n) \in D_1, \\ \dfrac{n}{n_1} Ty_{m_1,n} + \left(1 - \dfrac{n}{n_1}\right), & (m,n) \in D_2, \\ \dfrac{m}{m_1} Ty_{m,n_1} + \left(1 - \dfrac{m}{m_1}\right), & (m,n) \in D_3, \\ \dfrac{mn}{m_1 n_1} Ty_{m_1,n_1} + \left(1 - \dfrac{mn}{m_1 n_1}\right), & (m,n) \in D_4. \end{cases} \tag{2.592}$$

It is easy to see that $y_{m,n} > 0$ for $(m,n) \in D_2 \cup D_3 \cup D_4$ and hence $y_{m,n} > 0$ for all $(m,n) \in D_1$. Set

$$x_{m,n} = \lambda_{m,n} y_{m,n}, \tag{2.593}$$

then from (2.592) and (2.593), we have

$$x_{m,n} = \begin{cases} \dfrac{1}{d}\left\{\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(a-d)x_{m+1+i,n+1+i+j} + \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} bx_{m+i,n+1+i+j} \right. \\ \quad + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} cx_{m+1+i,n+i+j} + \sum_{j=0}^{\infty}(b-d)x_{m,n+1+j} \\ \quad \left. + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} P_{m+i,n+i+j}x_{m-k+i,n-l+i+j}\right\}, & (m,n) \in D_1, \\ \dfrac{n}{n_1}Ty_{m_1,n} + \left(1 - \dfrac{n}{n_1}\right), & (m,n) \in D_2, \\ \dfrac{m}{m_1}Ty_{m,n_1} + \left(1 - \dfrac{m}{m_1}\right), & (m,n) \in D_3, \\ \dfrac{mn}{m_1 n_1}Ty_{m_1,n_1} + \left(1 - \dfrac{mn}{m_1 n_1}\right), & (m,n) \in D_4. \end{cases} \tag{2.594}$$

And so

$$ax_{m+1,n+1} + bx_{m,n+1} + cx_{m+1,n} - dx_{m,n} + P_{m,n}x_{m-k,n-l} = 0, \quad (m,n) \in D_1, \tag{2.595}$$

which implies $x = \{x_{m,n}\}$ is a positive solution of (2.583). The proof is complete. □

Remark 2.113. Similar results for (2.583) have been obtained for the following cases: (i) $a \geq d$, $b \geq d$, $c < d$; (ii) $a \geq d$, $b < d$, $c \geq d$; (iii) $a \geq d$, $b < d$, $c < d$; (iv) $a < d$, $b \geq d$, $c \geq d$; (v) $a < d$, $b \geq d$, $c < d$; (vi) $a < d$, $b < d$, $c \geq d$; (vii) $a < d$, $b < d$, $c < d$.

2.11.2. Equation with neutral delay type

We consider the higher order partial difference equation of neutral type

$$\Delta_n^h \Delta_m^r (A_{m,n} + cA_{m-k,n-l}) + \sum_{s=1}^{u} p_{m,n}^{(s)} A_{m-\tau_s,n-\sigma_s} = f_{m,n}, \tag{2.596}$$

where $h, r, u \in N_1$, $k, l, \tau_s, \sigma_s \in N_0$, $c \in R$, and $p^{(s)}$, $f : N_{m_0} \times N_{n_0} \to R$, $s = 1, 2, \ldots, u$.

The higher order partial differences for any positive integers r and h are defined as $\Delta_m^r A_{m,n} = \Delta_m(\Delta_m^{r-1}A_{m,n})$, $\Delta_m^0 A_{m,n} = A_{m,n}$, $\Delta_n^h A_{m,n} = \Delta_n(\Delta_n^{h-1}A_{m,n})$, and $\Delta_n^0 A_{m,n} = A_{m,n}$. For $t \in R$ we define the usual factorial expression $(t)^{(m)} = t(t-1)\cdots(t-m+1)$ with $(t)^{(0)} = 1$.

Let $\delta = \max_{1\le s\le u}\{k,\tau_s\}$, $\eta = \max_{1\le s\le u}\{l,\sigma_s\}$ and $M_0 \ge m_0$, $N_0 \ge n_0$ be fixed nonnegative integers. By a solution of (2.596), we mean a nontrivial double sequence $\{A_{m,n}\}$ which is defined on $N_{m_0-\delta}\times N_{n_0-\eta}$ and satisfies (2.596) on $N_{m_0}\times N_{n_0}$.

In this section, we consider the existence of positive solutions of (2.596) in the case when $\{p_{m,n}^{(s)}\}$, $s=1,2,\dots,u$ and $\{f_{m,n}\}$ can change sign.

Theorem 2.114. *Assume that $c\ne -1$ and that*

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}(m)^{(r-1)}(n)^{(h-1)}\,|\,p_{m,n}^{(s)}\,| < \infty, \quad s=1,2,\dots,u,$$
$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}(m)^{(r-1)}(n)^{(h-1)}\,|\,f_{m,n}\,| < \infty. \tag{2.597}$$

Then (2.596) *has a bounded positive solution.*

Proof. The proof of this theorem will be divided into five cases in terms of c. Let X be the set of all real double sequence $A=\{A_{m,n}\}_{m=m_0,\,n=n_0}^{\infty\quad\infty}$ with the norm $\|A\| = \sup_{m\ge m_0,\,n\ge n_0}|A_{m,n}|<\infty$. X is a Banach space.

Case 1. For the case $-1<c\le 0$, choose $m_1>m_0$, $n_1>n_0$ sufficiently large such that $m_1-\max\{\delta,r\}\ge m_0$, $n_1-\max\{\eta,h\}\ge n_0$ and

$$\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\sum_{s=1}^{u}\,|\,p_{i,j}^{(s)}\,| \le \frac{1+c}{8},$$
$$\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\,|\,f_{i,j}\,| \le \frac{1+c}{6}. \tag{2.598}$$

Set

$$D=\mathbf{N}_{m_0}\times\mathbf{N}_{n_0}, \qquad D_1=\mathbf{N}_{m_1}\times\mathbf{N}_{n_1},$$
$$D_2=\mathbf{N}_{m_0}\times\mathbf{N}_{n_1}\setminus D_1, \qquad D_3=\mathbf{N}_{m_1}\times\mathbf{N}_{n_0}\setminus D_1, \tag{2.599}$$
$$D_4=D\setminus(D_1\cup D_2\cup D_3).$$

Clearly, $D=D_1\cup D_2\cup D_3\cup D_4$.

We define a closed, bounded, and convex subset Ω of X as follows:

$$\Omega = \left\{ A = \{A_{m,n}\} \in X \mid \frac{2(1+c)}{3} \le A_{m,n} \le \frac{4}{3},\ (m,n) \in D \right\}. \tag{2.600}$$

Define a mapping $T : \Omega \to X$ as follows:

$$TA_{m,n} = \begin{cases} 1 + c - cA_{m-k,n-l} + \dfrac{(-1)^{r+h+1}}{(r-1)!(h-1)!} \\ \quad \times \displaystyle\sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} \\ \quad \times \left(\displaystyle\sum_{s=1}^{u} p_{i,j}^{(s)} A_{i-\tau_s, j-\sigma_s} - f_{i,j} \right), & (m,n) \in D_1, \\ TA_{m_1,n}, & (m,n) \in D_2, \\ TA_{m,n_1}, & (m,n) \in D_3, \\ TA_{m_1,n_1}, & (m,n) \in D_4. \end{cases} \tag{2.601}$$

We will show that $T\Omega \subset \Omega$. In fact, for every $A \in \Omega$ and $m \ge m_1$, $n \ge n_1$, we get

$$\begin{aligned} TA_{m,n} &\le 1 + c - cA_{m-k,n-l} + \frac{1}{(r-1)!(h-1)!} \\ &\quad \times \sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} \\ &\quad \times \left(\sum_{s=1}^{u} \left| p_{i,j}^{(s)} \right| \left| A_{i-\tau_s, j-\sigma_s} \right| + \left| f_{i,j} \right| \right) \\ &\le 1 + c - \frac{4}{3}c + \frac{1}{(r-1)!(h-1)!} \sum_{i=m_1}^{\infty} (i)^{(r-1)} \sum_{j=n_1}^{\infty} (j)^{(h-1)} \\ &\quad \times \left(\frac{4}{3} \sum_{s=1}^{u} \left| p_{i,j}^{(s)} \right| + \left| f_{i,j} \right| \right) \\ &\le 1 + c - \frac{4}{3}c + \frac{4}{3}\frac{1+c}{8} + \frac{1+c}{6} = \frac{4}{3}. \end{aligned} \tag{2.602}$$

Furthermore, we have

$$\begin{aligned}
TA_{m,n} &\geq 1+c-cA_{m-k,n-l}-\frac{1}{(r-1)!(h-1)!}\\
&\quad\times\sum_{i=m}^{\infty}(i-m+r-1)^{(r-1)}\sum_{j=n}^{\infty}(j-n+h-1)^{(h-1)}\\
&\quad\times\left(\sum_{s=1}^{u}\left|p_{i,j}^{(s)}\right|\left|A_{i-\tau_s,j-\sigma_s}\right|+\left|f_{i,j}\right|\right)\\
&\geq 1+c-\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\\
&\quad\times\left(\frac{4}{3}\sum_{s=1}^{u}\left|p_{i,j}^{(s)}\right|+\left|f_{i,j}\right|\right)\\
&\geq 1+c-\frac{4}{3}\frac{1+c}{8}-\frac{1+c}{6}=\frac{2(1+c)}{3}.
\end{aligned}\tag{2.603}$$

Hence,

$$\frac{2(1+c)}{3}\leq TA_{m,n}\leq\frac{4}{3}\quad\text{for }(m,n)\in D.\tag{2.604}$$

Thus, we have $T\Omega\subset\Omega$.

Now, we claim that T is a contraction mapping on Ω. In fact, for $B,A\in\Omega$ and $(m,n)\in D_1$, we have

$$\begin{aligned}
&\left|TB_{m,n}-TA_{m,n}\right|\\
&\quad\leq -c\left|B_{m-k,n-l}-A_{m-k,n-l}\right|\\
&\qquad+\frac{1}{(r-1)!(h-1)!}\sum_{i=m}^{\infty}(i-m+r-1)^{(r-1)}\sum_{j=n}^{\infty}(j-n+h-1)^{(h-1)}\\
&\qquad\times\sum_{s=1}^{u}\left|p_{i,j}^{(s)}\right|\left|B_{i-\tau_s,j-\sigma_s}-A_{i-\tau_s,j-\sigma_s}\right|\\
&\quad\leq -c\left|B_{m-k,n-l}-A_{m-k,n-l}\right|\\
&\qquad+\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\\
&\qquad\times\sum_{s=1}^{u}\left|p_{i,j}^{(s)}\right|\left|B_{i-\tau_s,j-\sigma_s}-A_{i-\tau_s,j-\sigma_s}\right|\\
&\quad\leq\frac{1-7c}{8}\|B-A\|.
\end{aligned}\tag{2.605}$$

This implies that

$$\|TB - TA\| \leq \frac{1-7c}{8}\|B - A\|. \tag{2.606}$$

Since $0 < (1-7c)/8 < 1$, T is a contraction mapping on Ω. Therefore, by the Banach contraction mapping principle, T has a fixed point A^0 in Ω, that is, $TA^0 = A^0$. Clearly, $A^0 = \{A^0_{m,n}\}$ is a bounded positive solution of (2.596). This completes the proof in this case.

Case 2. For the case $c < -1$, choose $m_1 > m_0$, $n_1 > n_0$ sufficiently large so that $m_1 - \max\{\delta, r\} \geq m_0$, $n_1 - \max\{\eta, h\} \geq n_0$ and

$$\begin{gathered}\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\sum_{s=1}^{u}|p^{(s)}_{i,j}| \leq -\frac{1+c}{8},\\ \frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}|f_{i,j}| \leq \frac{c(1+c)}{4}.\end{gathered} \tag{2.607}$$

We define a closed, bounded, and convex subset Ω of X as follows:

$$\Omega = \left\{A = \{A_{m,n}\} \in X \mid -\frac{c}{2} \leq A_{m,n} \leq -2c,\ (m,n) \in D\right\}. \tag{2.608}$$

Define a mapping $T : \Omega \to X$ as follows:

$$TA_{m,n} = \begin{cases} -c - 1 - \dfrac{1}{c}A_{m+k,n+l} + \dfrac{(-1)^{r+h+1}}{c(r-1)!(h-1)!} \\ \quad\times \displaystyle\sum_{i=m+k}^{\infty}(i-m-k+r-1)^{(r-1)} \\ \quad\times \displaystyle\sum_{j=n+l}^{\infty}(j-n-l+h-1)^{(h-1)} \\ \quad\times \left(\displaystyle\sum_{s=1}^{u}p^{(s)}_{i,j}A_{i-\tau_s,j-\sigma_s} - f_{i,j}\right), & (m,n) \in D_1, \\ TA_{m_1,n}, & (m,n) \in D_2, \\ TA_{m,n_1}, & (m,n) \in D_3, \\ TA_{m_1,n_1}, & (m,n) \in D_4. \end{cases} \tag{2.609}$$

We will show that $T\Omega \subset \Omega$. In fact, for every $A \in \Omega$ and $(m,n) \in D_1$, we get

$$\begin{aligned}
TA_{m,n} &\leq -c - 1 - \frac{1}{c}A_{m+k,n+l} - \frac{1}{c(r-1)!(h-1)!} \\
&\quad \times \sum_{i=m+k}^{\infty}(i-m-k+r-1)^{(r-1)}\sum_{j=n+l}^{\infty}(j-n-l+h-1)^{(h-1)} \\
&\quad \times \left(\sum_{s=1}^{u}|p_{i,j}^{(s)}||A_{i-\tau_s,j-\sigma_s}| + |f_{i,j}|\right) \\
&\leq -c - 1 + 2 - \frac{1}{c(r-1)!(h-1)!}\sum_{i=m_1+k}^{\infty}(i)^{(r-1)}\sum_{j=n_1+l}^{\infty}(j)^{(h-1)} \\
&\quad \times \left(-2c\sum_{s=1}^{u}|p_{i,j}^{(s)}| + |f_{i,j}|\right) \\
&\leq -c + 1 - \frac{1}{c}\left(\frac{c(1+c)}{4} + \frac{c(1+c)}{4}\right) \leq -2c.
\end{aligned} \tag{2.610}$$

Furthermore, we have

$$\begin{aligned}
TA_{m,n} &\geq -c - 1 - \frac{1}{c}A_{m+k,n+l} + \frac{1}{c(r-1)!(h-1)!} \\
&\quad \times \sum_{i=m+k}^{\infty}(i-m-k+r-1)^{(r-1)}\sum_{j=n+l}^{\infty}(j-n-l+h-1)^{(h-1)} \\
&\quad \times \left(\sum_{s=1}^{u}|p_{i,j}^{(s)}||A_{i-\tau_s,j-\sigma_s}| + |f_{i,j}|\right) \\
&\geq -c - 1 - \frac{1}{c}\left(-\frac{c}{2}\right) + \frac{1}{c(r-1)!(h-1)!} \\
&\quad \times \sum_{i=m_1+k}^{\infty}(i)^{(r-1)}\sum_{j=n_1+l}^{\infty}(j)^{(h-1)}\left(-2c\sum_{s=1}^{u}|p_{i,j}^{(s)}| + |f_{i,j}|\right) \\
&\geq -c - \frac{1}{2} + \frac{1}{c}\left(\frac{c(1+c)}{4} + \frac{c(1+c)}{4}\right) = -\frac{c}{2}.
\end{aligned} \tag{2.611}$$

Hence,

$$-\frac{c}{2} \leq TA_{m,n} \leq -2c \quad \text{for } (m,n) \in D. \tag{2.612}$$

Thus we have proved that $T\Omega \subset \Omega$.

Now, we will show that T is a contraction mapping on Ω. In fact, for $B, A \in \Omega$, and $(m,n) \in D_1$, we have

$$\begin{aligned}
&|TB_{m,n} - TA_{m,n}| \\
&\quad \le -\frac{1}{c}|B_{m+k,n+l} - A_{m+k,n+l}| \\
&\qquad - \frac{1}{c(r-1)!(h-1)!}\sum_{i=m+k}^{\infty}(i-m-k+r-1)^{(r-1)} \\
&\qquad \times \sum_{j=n+l}^{\infty}(j-n-l+h-1)^{(h-1)}\sum_{s=1}^{u}|p_{i,j}^{(s)}|\,|B_{i-\tau_s,j-\sigma_s} - A_{i-\tau_s,j-\sigma_s}| \\
&\quad \le -\frac{1}{c}|B_{m+k,n+l} - A_{m+k,n+l}| \\
&\qquad - \frac{1}{c(r-1)!(h-1)!}\sum_{i=m_1+k}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)} \\
&\qquad \times \sum_{s=1}^{u}|p_{i,j}^{(s)}|\,|B_{i-\tau_s,j-\sigma_s} - A_{i-\tau_s,j-\sigma_s}| \\
&\quad \le \frac{c-7}{8c}\|B-A\|.
\end{aligned} \tag{2.613}$$

This implies that

$$\|TB - TA\| \le \frac{c-7}{8c}\|B-A\|. \tag{2.614}$$

Since $0 < (c-7)/8c < 1$, so T is a contraction mapping on Ω. Therefore, by the Banach contraction mapping principle, T has a fixed point A^0 in Ω, that is, $TA^0 = A^0$. Clearly, $A^0 = \{A^0_{m,n}\}$ is a bounded positive solution of (2.596). This completes the proof in this case.

Case 3. For the case $0 \le c < 1$, choose $m_1 > m_0$, $n_1 > n_0$ sufficiently large such that $m_1 - \max\{\delta, r\} \ge m_0$, $n_1 - \max\{\eta, h\} \ge n_0$ and

$$\begin{gathered}
\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\sum_{s=1}^{u}|p_{i,j}^{(s)}| \le \frac{1-c}{8}, \\
\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}|f_{i,j}| \le \frac{1-c}{2}.
\end{gathered} \tag{2.615}$$

We define a closed, bounded, and convex subset Ω of X as follows:

$$\Omega = \{A = \{A_{m,n}\} \in X \mid 2(1-c) \le A_{m,n} \le 4,\ (m,n) \in D\}. \tag{2.616}$$

Define a mapping $T : \Omega \to X$ as follows:

$$TA_{m,n} = \begin{cases} 3 + c - cA_{m-k,n-l} + \dfrac{(-1)^{r+h+1}}{(r-1)!(h-1)!} \\ \quad \times \displaystyle\sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} \\ \quad \times \left(\displaystyle\sum_{s=1}^{u} p_{i,j}^{(s)} A_{i-\tau_s, j-\sigma_s} - f_{i,j} \right), & (m,n) \in D_1, \\ TA_{m_1,n}, & (m,n) \in D_2, \\ TA_{m,n_1}, & (m,n) \in D_3, \\ TA_{m_1,n_1}, & (m,n) \in D_4. \end{cases} \tag{2.617}$$

We will show that $T\Omega \subset \Omega$. In fact, for every $A \in \Omega$ and $(m,n) \in D_1$, we get

$$\begin{aligned} TA_{m,n} &\le 3 + c - cA_{m-k,n-l} \\ &\quad + \frac{1}{(r-1)!(h-1)!} \sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \\ &\quad \times \sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} \left(\sum_{s=1}^{u} |p_{i,j}^{(s)}| \, |A_{i-\tau_s, j-\sigma_s}| + |f_{i,j}| \right) \\ &\le 3 + c + \frac{1}{(r-1)!(h-1)!} \sum_{i=m_1}^{\infty} (i)^{(r-1)} \sum_{j=n_1}^{\infty} (j)^{(h-1)} \\ &\quad \times \left(4 \sum_{s=1}^{u} |p_{i,j}^{(s)}| + |f_{i,j}| \right) \\ &\le 3 + c + 4\frac{1-c}{8} + \frac{1-c}{2} = 4. \end{aligned} \tag{2.618}$$

Furthermore, we have

$$
\begin{aligned}
TA_{m,n} &\geq 3 + c - cA_{m-k,n-l} \\
&\quad - \frac{1}{(r-1)!(h-1)!} \sum_{i=m}^{\infty} (i - m + r - 1)^{(r-1)} \\
&\quad \times \sum_{j=n}^{\infty} (j - n + h - 1)^{(h-1)} \left(\sum_{s=1}^{u} \left| p_{i,j}^{(s)} \right| \left| A_{i-\tau_s, j-\sigma_s} \right| + \left| f_{i,j} \right| \right) \\
&\geq 3 + c - 4c - \frac{1}{(r-1)!(h-1)!} \sum_{i=m_1}^{\infty} (i)^{(r-1)} \sum_{j=n_1}^{\infty} (j)^{(h-1)} \\
&\quad \times \left(4 \sum_{s=1}^{u} \left| p_{i,j}^{(s)} \right| + \left| f_{i,j} \right| \right) \\
&\geq 3 + c - 4c - 4\frac{1-c}{8} - \frac{1-c}{2} = 2(1-c).
\end{aligned}
\tag{2.619}
$$

Hence,

$$
2(1-c) \leq TA_{m,n} \leq 4 \quad \text{for } (m,n) \in D. \tag{2.620}
$$

Thus we have proved that $T\Omega \subset \Omega$.

Now, we will show that T is a contraction mapping on Ω. In fact, for $B, A \in \Omega$ and $m \geq m_1,\ n \geq n_1$, we have

$$
\begin{aligned}
&\left| TB_{m,n} - TA_{m,n} \right| \\
&\quad \leq c \left| B_{m-k,n-l} - A_{m-k,n-l} \right| \\
&\qquad + \frac{1}{(r-1)!(h-1)!} \sum_{i=m}^{\infty} (i - m + r - 1)^{(r-1)} \\
&\qquad \times \sum_{j=n}^{\infty} (j - n + h - 1)^{(h-1)} \sum_{s=1}^{u} \left| p_{i,j}^{(s)} \right| \left| B_{i-\tau_s, j-\sigma_s} - A_{i-\tau_s, j-\sigma_s} \right| \\
&\quad \leq c \left| B_{m-k,n-l} - A_{m-k,n-l} \right| \\
&\qquad + \frac{1}{(r-1)!(h-1)!} \sum_{i=m_1}^{\infty} (i)^{(r-1)} \sum_{j=n_1}^{\infty} (j)^{(h-1)} \\
&\qquad \times \sum_{s=1}^{u} \left| p_{i,j}^{(s)} \right| \left| B_{i-\tau_s, j-\sigma_s} - A_{i-\tau_s, j-\sigma_s} \right| \\
&\quad \leq \frac{1+7c}{8} \| B - A \|.
\end{aligned}
\tag{2.621}
$$

This implies that

$$\|TB - TA\| \leq \frac{1+7c}{8}\|B - A\|. \tag{2.622}$$

Since $0 < (1+7c)/8 < 1$, T is a contraction mapping on Ω. Therefore, by the Banach contraction mapping principle, T has a fixed point A^0 in Ω, that is, $TA^0 = A^0$. Clearly, $A^0 = \{A^0_{m,n}\}$ is a bounded positive solution of (2.596). This completes the proof in this case.

Case 4. For the case $c > 1$, choose $m_1 > m_0$, $n_1 > n_0$ sufficiently large such that $m_1 - \max\{\delta, r\} \geq m_0$, $n_1 - \max\{\eta, h\} \geq n_0$ and

$$\begin{gathered}\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\sum_{s=1}^{u}\left|p^{(s)}_{i,j}\right| \leq \frac{c-1}{8},\\ \frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\left|f_{i,j}\right| \leq \frac{c-1}{2}.\end{gathered} \tag{2.623}$$

We define a closed, bounded, and convex subset Ω of X as follows:

$$\Omega = \left\{A = \{A_{m,n}\} \in X \mid 2(c-1) \leq A_{m,n} \leq 4c,\ (m,n) \in D\right\}. \tag{2.624}$$

Define a mapping $T : \Omega \to X$ as follows:

$$TA_{m,n} = \begin{cases} 3c + 1 - \dfrac{1}{c}A_{m+k,n+l} + \dfrac{(-1)^{r+h+1}}{c(r-1)!(h-1)!} & \\ \quad\times \displaystyle\sum_{i=m+k}^{\infty}(i-m-k+r-1)^{(r-1)} & \\ \quad\times \displaystyle\sum_{j=n+l}^{\infty}(j-n-l+h-1)^{(h-1)} & \\ \quad\times\left(\displaystyle\sum_{s=1}^{u}p^{(s)}_{i,j}A_{i-\tau_s,j-\sigma_s} - f_{i,j}\right), & (m,n) \in D_1,\\ TA_{m_1,n}, & (m,n) \in D_2,\\ TA_{m,n_1}, & (m,n) \in D_3,\\ TA_{m_1,n_1}, & (m,n) \in D_4. \end{cases} \tag{2.625}$$

We will show that $T\Omega \subset \Omega$. In fact, for every $A \in \Omega$ and $(m,n) \in D_1$, we get

$$\begin{aligned}
TA_{m,n} &\leq 3c + 1 - \frac{1}{c}A_{m+k,n+l} \\
&\quad + \frac{1}{c(r-1)!(h-1)!}\sum_{i=m+k}^{\infty}(i-m-k+r-1)^{(r-1)} \\
&\quad \times \sum_{j=n+l}^{\infty}(j-n-l+h-1)^{(h-1)}\left(\sum_{s=1}^{u}\left|p_{i,j}^{(s)}\right|\left|A_{i-\tau_s,j-\sigma_s}\right| + \left|f_{i,j}\right|\right) \\
&\leq 3c + 1 + \frac{1}{c(r-1)!(h-1)!}\sum_{i=m_1+k}^{\infty}(i)^{(r-1)}\sum_{j=n_1+l}^{\infty}(j)^{(h-1)} \\
&\quad \times \left(4c\sum_{s=1}^{u}\left|p_{i,j}^{(s)}\right| + \left|f_{i,j}\right|\right) \\
&\leq 3c + 1 + 4c\frac{c-1}{8c} + \frac{c-1}{2} = 4c.
\end{aligned} \tag{2.626}$$

Furthermore, we have

$$\begin{aligned}
TA_{m,n} &\geq 3c + 1 - \frac{1}{c}A_{m+k,n+l} \\
&\quad - \frac{1}{c(r-1)!(h-1)!}\sum_{i=m+k}^{\infty}(i-m-k+r-1)^{(r-1)} \\
&\quad \times \sum_{j=n+l}^{\infty}(j-n-l+h-1)^{(h-1)}\left(\sum_{s=1}^{u}\left|p_{i,j}^{(s)}\right|\left|A_{i-\tau_s,j-\sigma_s}\right| + \left|f_{i,j}\right|\right) \\
&\geq 3c + 1 - 4 - \frac{1}{c(r-1)!(h-1)!}\sum_{i=m_1+k}^{\infty}(i)^{(r-1)}\sum_{j=n_1+l}^{\infty}(j)^{(h-1)} \\
&\quad \times \left(4c\sum_{s=1}^{u}\left|p_{i,j}^{(s)}\right| + \left|f_{i,j}\right|\right) \\
&\geq 3c - 3 - 4c\frac{c-1}{8c} - \frac{c-1}{2} = 2(c-1).
\end{aligned} \tag{2.627}$$

Hence,

$$2(c-1) \leq TA_{m,n} \leq 4c \quad \text{for } (m,n) \in D. \tag{2.628}$$

Thus we have proved that $T\Omega \subset \Omega$.

Now, we will show that T is a contraction mapping on Ω. In fact, for $B, A \in \Omega$, and $(m,n) \in D_1$, we can prove that

$$\left| TB_{m,n} - TA_{m,n} \right| \le \frac{c+7}{8c} \|B - A\|. \tag{2.629}$$

This implies that

$$\|TB - TA\| \le \frac{c+7}{8c} \|B - A\|. \tag{2.630}$$

Since $0 < (c+7)/8c < 1$, T is a contraction mapping on Ω. Therefore, by the Banach contraction mapping principle, T has a fixed point A^0 in Ω, that is, $TA^0 = A^0$. Clearly, $A^0 = \{A^0_{m,n}\}$ is a bounded positive solution of (2.596). This completes the proof in this case.

Case 5. Finally, we consider the last case when $c = 1$. Let $m_1 > m_0$, $n_1 > n_0$ be such that $m_1 - \max\{\delta, r\} \ge m_0$, $n_1 - \max\{\eta, h\} \ge n_0$ and

$$\begin{gathered}
\frac{1}{(r-1)!(h-1)!} \sum_{i=m_1+k}^{\infty} (i)^{(r-1)} \sum_{j=n_1+l}^{\infty} (j)^{(h-1)} \sum_{s=1}^{u} \left| p_{i,j}^{(s)} \right| \le \frac{1}{8}, \\
\frac{1}{(r-1)!(h-1)!} \sum_{i=m_1+k}^{\infty} (i)^{(r-1)} \sum_{j=n_1+l}^{\infty} (j)^{(h-1)} \left| f_{i,j} \right| \le \frac{1}{2}.
\end{gathered} \tag{2.631}$$

We define a closed, bounded, and convex subset Ω of X as follows:

$$\Omega = \left\{ A = \left\{ A_{m,n} \right\} \in X \mid 2 \le A_{m,n} \le 4,\ (m,n) \in D \right\}. \tag{2.632}$$

Define a mapping $T : \Omega \to X$ as follows:

$$TA_{m,n} = \begin{cases}
3 + \dfrac{(-1)^{r+h+1}}{(r-1)!(h-1)!} \displaystyle\sum_{w=1}^{\infty} \sum_{i=m+(2w-1)k}^{m+2wk-1} (i-m+r-1)^{(r-1)} \\
\quad \times \displaystyle\sum_{v=1}^{\infty} \sum_{j=n+(2v-1)l}^{n+2vl-1} (j-n+h-1)^{(h-1)} \\
\quad \times \left(\displaystyle\sum_{s=1}^{u} p_{i,j}^{(s)} A_{i-\tau,j-\sigma} - f_{i,j} \right), & (m,n) \in D_1, \\
TA_{m_1,n}, & (m,n) \in D_2, \\
TA_{m,n_1}, & (m,n) \in D_3, \\
TA_{m_1,n_1}, & (m,n) \in D_4.
\end{cases} \tag{2.633}$$

By a similar argument to that of Cases 1–4, we can easily show that T maps Ω into Ω and for $B, A \in \Omega$,

$$\|TB - TA\| \leq \frac{1}{8}\|B - A\|. \tag{2.634}$$

Therefore, by the Banach contraction principle, T has a fixed point A^0 in Ω, that is,

$$A^0_{m,n} = \begin{cases} 3 + \dfrac{(-1)^{r+h+1}}{(r-1)!(h-1)!} \displaystyle\sum_{w=1}^{\infty} \sum_{i=m+(2w-1)k}^{m+2wk-1} (i-m+r-1)^{(r-1)} \\ \quad \times \displaystyle\sum_{v=1}^{\infty} \sum_{j=n+(2v-1)l}^{n+2vl-1} (j-n+h-1)^{(h-1)} \\ \quad \times \left(\displaystyle\sum_{s=1}^{u} p^{(s)}_{i,j} A^0_{i-\tau,j-\sigma} - f_{i,j} \right), & (m,n) \in D_1, \\ A^0_{m_1,n}, & (m,n) \in D_2, \\ A^0_{m,n_1}, & (m,n) \in D_3, \\ A^0_{m_1,n_1}, & (m,n) \in D_4. \end{cases} \tag{2.635}$$

It follows that for $(m,n) \in D_1$

$$\begin{aligned} A^0_{m,n} + A^0_{m-k,n-l} = 6 &+ \frac{(-1)^{r+h+1}}{(r-1)!(h-1)!} \sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \\ &\times \sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} \left(\sum_{s=1}^{u} p^{(s)}_{i,j} A^0_{i-\tau,j-\sigma} - f_{i,j} \right). \end{aligned} \tag{2.636}$$

Clearly, $A^0 = \{A^0_{m,n}\}$ is a bounded positive solution of (2.596). This completes the proof of Theorem 2.114. □

Example 2.115. Consider the higher order neutral partial difference equation

$$\Delta^h_n \Delta^r_m (A_{m,n} + cA_{m-k,n-l}) + \frac{1}{m^\alpha n^\beta} A_{m-\tau,n-\sigma} = 0, \tag{2.637}$$

where r, h, k, l, τ, and σ are positive integers, $c \in R$, $\alpha, \beta \in R^+$ and $\alpha > r$, $\beta > h$. Since

$$\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} (m)^{(r-1)}(n)^{(h-1)} \frac{1}{m^\alpha n^\beta} \le \sum_{m=m_0}^{\infty} \frac{1}{m^{\alpha+1-r}} \sum_{n=n_0}^{\infty} \frac{1}{n^{\beta+1-h}} < \infty, \tag{2.638}$$

by Theorem 2.114, (2.637) has a bounded positive solution.

Theorem 2.116. *Assume that* $c = -1$ *and that*

$$\begin{gathered} \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} m(m)^{(r-1)} n(n)^{(h-1)} \left| p_{m,n}^{(s)} \right| < \infty, \quad s = 1, 2, \dots, u, \\ \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} m(m)^{(r-1)} n(n)^{(h-1)} \left| f_{m,n} \right| < \infty. \end{gathered} \tag{2.639}$$

Then (2.596) *has a bounded positive solution.*

Proof. By a known result [58], (2.639) are equivalent to

$$\begin{gathered} \sum_{w=0}^{\infty} \sum_{u=0}^{\infty} \sum_{m=m_0+wk}^{\infty} \sum_{n=n_0+ul}^{\infty} (m)^{(r-1)}(n)^{(h-1)} \left| p_{m,n}^{(s)} \right| < \infty, \quad s = 1, 2, \dots, u, \\ \sum_{w=0}^{\infty} \sum_{u=0}^{\infty} \sum_{m=m_0+wk}^{\infty} \sum_{n=n_0+ul}^{\infty} (m)^{(r-1)}(n)^{(h-1)} \left| f_{m,n} \right| < \infty, \end{gathered} \tag{2.640}$$

respectively. We choose sufficiently large $m_1 > m_0$, $n_1 > n_0$ such that $m_1 - \max\{\delta, r\} \ge m_0$, $n_1 - \max\{\eta, h\} \ge n_0$ and

$$\begin{gathered} \frac{1}{(r-1)!(h-1)!} \sum_{w=0}^{\infty} \sum_{u=0}^{\infty} \sum_{i=m_1+wk}^{\infty} (i)^{(r-1)} \sum_{j=n_1+ul}^{\infty} (j)^{(h-1)} \sum_{s=1}^{u} \left| p_{i,j}^{(s)} \right| \le \frac{1}{8}, \\ \frac{1}{(r-1)!(h-1)!} \sum_{w=0}^{\infty} \sum_{u=0}^{\infty} \sum_{i=m_1+wk}^{\infty} (i)^{(r-1)} \sum_{j=n_1+ul}^{\infty} (j)^{(h-1)} \left| f_{i,j} \right| \le \frac{1}{2}. \end{gathered} \tag{2.641}$$

We define a closed, bounded, and convex subset Ω of X as follows:

$$\Omega = \left\{ A = \{A_{m,n}\} \in X \mid 2 \le A_{m,n} \le 4,\ (m,n) \in D \right\}. \tag{2.642}$$

Define a mapping $T : \Omega \to X$ as follows:

$$TA_{m,n} = \begin{cases} 3 + \dfrac{(-1)^{r+h}}{(r-1)!(h-1)!} \displaystyle\sum_{w=1}^{\infty} \sum_{i=m+wk}^{\infty} (i-m+r-1)^{(r-1)} \\ \quad \times \displaystyle\sum_{u=1}^{\infty} \sum_{j=n+ul}^{\infty} (j-n+h-1)^{(h-1)} \\ \quad \times \left(\displaystyle\sum_{s=1}^{u} p_{i,j}^{(s)} A_{i-\tau,j-\sigma} - f_{i,j} \right), & (m,n) \in D_1, \\ TA_{m_1,n}, & (m,n) \in D_2, \\ TA_{m,n_1}, & (m,n) \in D_3, \\ TA_{m_1,n_1}, & (m,n) \in D_4. \end{cases} \tag{2.643}$$

By a similar argument to that of Cases 1–5 in Theorem 2.114, we can easily show that T maps Ω into Ω and for $B, A \in \Omega$,

$$\|TB - TA\| \leq \frac{1}{8} \|B - A\|. \tag{2.644}$$

Therefore, by the Banach contraction principle, T has a fixed point A^0 in Ω, that is,

$$A_{m,n}^0 = \begin{cases} 3 + \dfrac{(-1)^{r+h}}{(r-1)!(h-1)!} \displaystyle\sum_{w=1}^{\infty} \sum_{i=m+wk}^{\infty} (i-m+r-1)^{(r-1)} \\ \quad \times \displaystyle\sum_{u=1}^{\infty} \sum_{j=n+ul}^{\infty} (j-n+h-1)^{(h-1)} \\ \quad \times \left(\displaystyle\sum_{s=1}^{u} p_{i,j}^{(s)} A_{i-\tau,j-\sigma}^0 - f_{i,j} \right), & (m,n) \in D_1, \\ TA_{m_1,n}^0, & (m,n) \in D_2, \\ TA_{m,n_1}^0, & (m,n) \in D_3, \\ TA_{m_1,n_1}^0, & (m,n) \in D_4. \end{cases} \tag{2.645}$$

It follows that for $(m,n) \in D_1$

$$\begin{aligned} A_{m,n}^0 - A_{m-k,n-l}^0 = 6 + \frac{(-1)^{r+h+1}}{(r-1)!(h-1)!} \sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \\ \times \sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} \left(\sum_{s=1}^{u} p_{i,j}^{(s)} A_{i-\tau,j-\sigma}^0 - f_{i,j} \right). \end{aligned} \tag{2.646}$$

Clearly, $A^0 = \{A^0_{m,n}\}$ is a bounded positive solution of (2.596). This completes the proof of Theorem 2.116. □

2.12. Notes

The material of Section 2.2 is based on Zhang and Liu [171, 172]. The related work can be seen from Liu and Wang [105]. The results in Section 2.3 are taken from Zhang and Liu [174]. The material of Section 2.4 is based on Zhang and Liu [167]. The method in Section 2.5.1 is presented first in Zhang and Liu [173]. The material of Section 2.5.1 is based on Choi and Zhang [46]. The results in Section 2.5.2 are taken from Zhang and Tian [178]. The material of Section 2.5.3 is taken from Agarwal and Zhou [7], the related papers can be seen from Zhang [161], Choi et al. [45], Cui and Liu [50]. The results in Section 2.6.1 are adopted from Zhang and Zhou [187]. The material of Section 2.6.2 is new [188]. Section 2.6.3 is taken from Zhang and Liu [170]. The concept of frequent oscillation is posed by Tian et al. [136]. The material of Section 2.7 is taken from Tian and Zhang [141], the related work, see Xie and Tian [156]. The material of Section 2.8 is based on Xie et al. [158]. The material of Section 2.9 is taken from Liu and Zhang [107], the related work, see Liu et al. [104], Liu et al. [108]. The material of Section 2.10 is taken from Zhang and Zhou [189]. Theorem 2.112 is taken from Zhang and Xing [182]. In [182], authors discuss the various cases in Remark 2.113 and present another method to study the existence of positive solutions of (2.583). Theorem 2.114 is based on Zhou et al. [193]. Theorem 2.116 is new.

3 Oscillations of nonlinear delay partial difference equations

3.1. Introduction

Nonlinear PDEs are very important in applications. Many phenomena in biological, physical, and engineering sciences can be described by nonlinear equations. First, we consider a class of nonlinear PDEs with the almost linear property. We present the linearized oscillation theory in Section 3.2, which is similar to the well-known linearized stability theory in ODEs. In Section 3.3, we present some results for nonlinear PDEs with variable coefficients. In Section 3.4, we state the existence of oscillatory solutions for certain nonlinear PDEs. In Section 3.5, we consider the existence of positive solutions for certain nonlinear PDEs. In Section 3.6, we study some population models using the results in the former sections. In Section 3.7, we consider the oscillation of initial boundary value problems of PDEs, which are discrete analogs of the corresponding initial boundary value problems of partial differential equations. Average techniques are very effective for this case. In Section 3.8, we consider the oscillation of multidimensional IBVPs.

3.2. Linearized oscillations

3.2.1. Linearized oscillation for $A_{m+1,n} + A_{m,n+1} - pA_{m,n} + q_{m,n} f(x_{m-k,n-l}) = 0$

In Chapter 2, the linear delay partial difference equations

$$x_{m+1,n} + x_{m,n+1} - px_{m,n} + qx_{m-k,n-l} = 0, \quad (m,n) \in N_0^2, \tag{3.1}$$

have been investigated and various properties related to the oscillatory behavior of their solutions have been reported. The purpose of this section is to establish some connections between (3.1) and a more general nonlinear delay partial difference equation.

Consider the nonlinear functional inequality of the form

$$x_{m+1,n} + x_{m,n+1} - px_{m,n} + q_{m,n} f(x_{m-k,n-l}) \le 0, \quad (m,n) \in N_0^2, \tag{3.2}$$

and the associated nonlinear partial difference equation

$$x_{m+1,n} + x_{m,n+1} - p x_{m,n} + q_{m,n} f(x_{m-k,n-l}) = 0, \quad (m,n) \in N_0^2. \tag{3.3}$$

In (3.2) and (3.3), the numbers p, k, l, the sequence $\{q_{m,n}\}$, and the function f will be restricted by appropriate conditions. For now, we will assume through out this section that p is a positive number, k and l nonnegative integers such that $\min(k,l) > 0$, $\{q_{m,n}\}_{(m,n)\in N_0^2}$ a real double sequence, and f a real-valued function defined on R. By a solution of (3.2) or (3.3), we mean a real double sequence $x = \{x_{m,n}\}$, $m \geq -k$, $n \geq -l$, which satisfies (3.2) or (3.3). It is not difficult to formulate and prove an existence theorem for the solutions of (3.3) when appropriate initial conditions are given (e.g., see Chapter 1). As is customary, we say that a solution $x = \{x_{m,n}\}$ of (3.3) is eventually positive (eventually negative) if $x_{m,n} > 0$ (resp., $x_{m,n} < 0$) for all large m and all large n, and is oscillatory if it is neither eventually positive nor eventually negative.

First of all, we will establish a comparison theorem.

Theorem 3.1. *Suppose that $\overline{p}$ and p are real numbers such that $1 \geq \overline{p} \geq p > 0$. Suppose that $\{\overline{q}_{m,n}\}$ and $\{q_{m,n}\}$ are nonnegative sequences which satisfy $q_{m,n} \geq \overline{q}_{m,n} > 0$ for all large m and n. Suppose further that the functions $f, \overline{f} : R \to R$ satisfy $0 < \overline{f}(x) \leq f(x)$ for $x > 0$. If (3.2) has an eventually positive solution, then so does the following equation:*

$$x_{m+1,n} + x_{m,n+1} - \overline{p} x_{m,n} + \overline{q}_{m,n} \overline{f}(x_{m-k,n-l}) = 0, \quad (m,n) \in N_0^2. \tag{3.4}$$

Proof. Let $x = \{x_{m,n}\}$ be an eventually positive solution of (3.2) such that $x_{m,n} > 0$ for $m \geq M - k \geq 0$ and $n \geq N - l \geq 0$. Suppose further that $f(t) > 0$ for $t > 0$. Then summing (3.2) with respect to the second independent variable from n to ∞, we obtain

$$\sum_{j=n}^{\infty} x_{m+1,j} + (1-p)\sum_{j=n}^{\infty} x_{m,j+1} + p\sum_{j=n}^{\infty}(x_{m,j+1} - x_{m,j}) + \sum_{j=n}^{\infty} q_{m,j} f(x_{m-k,j-l}) \leq 0 \tag{3.5}$$

so that

$$\begin{aligned}
&\sum_{j=n+1}^{\infty} x_{m+1,j} + p(x_{m+1,n} - x_{m,n}) \\
&\qquad + (1-p)x_{m+1,n} + (1-p)\sum_{j=n}^{\infty} x_{m,j+1} + \sum_{j=n}^{\infty} q_{m,j} f(x_{m-k,j-l}) \leq 0.
\end{aligned} \tag{3.6}$$

Summing the above inequality with respect to the first independent variable from m to ∞, we obtain

$$\sum_{(i,j)=(m,n+1)}^{\infty} x_{i+1,j} + \sum_{(i,j)=(m,n)}^{\infty} q_{i,j} f(x_{i-k,j-l}) + (1-p)\left\{\sum_{i=m}^{\infty} x_{i+1,n} + \sum_{(i,j)=(m,n)}^{\infty} x_{i,j+1}\right\} \le p x_{m,n}. \tag{3.7}$$

Thus

$$x_{m,n} \ge \frac{1}{p}\left\{\sum_{(i,j)=(m,n+1)}^{\infty} x_{i+1,j} + \sum_{(i,j)=(m,n)}^{\infty} q_{i,j} f(x_{i-k,j-l})\right\} + \frac{1-p}{p}\left\{\sum_{i=m}^{\infty} x_{i+1,n} + \sum_{(i,j)=(m,n)}^{\infty} x_{i,j+1}\right\} \tag{3.8}$$

for $m \ge M$ and $n \ge N$. Let $\overline{p}$ be a real number such that $1 \ge \overline{p} \ge p$, let $\{\overline{q}_{m,n}\}_{(m,n)\in N_0^2}$ be a nonnegative sequence such that $q_{m,n} \ge \overline{q}_{m,n}$ for $(m,n) \in Z^2$, and further let $\overline{f}$ be a real and nondecreasing function defined on R satisfying $\overline{f}(x) \le f(x)$ for $x > 0$. Let Ω be the set of all real double sequences of the form $y = \{y_{m,n} \mid m \ge M-k,\ n \ge N-l\}$. Define an operator $T : \Omega \to \Omega$ by

$$(Ty)_{m,n} = \frac{1}{\overline{p}x_{m,n}}\left\{\sum_{(i,j)=(m,n+1)}^{\infty} x_{i+1,j}y_{i+1,j} + \sum_{(i,j)=(m,n)}^{\infty} \overline{q}_{i,j}\overline{f}(x_{i-k,j-l}y_{i-k,j-l})\right\} + \frac{1-\overline{p}}{\overline{p}x_{m,n}}\left\{\sum_{i=m}^{\infty} x_{i+1,n}y_{i+1,n} + \sum_{(i,j)=(m,n)}^{\infty} x_{i,j+1}y_{i,j+1}\right\} \tag{3.9}$$

for $m \ge M$ and $n \ge N$, and

$$(Ty)_{m,n} = 1 \tag{3.10}$$

elsewhere. Consider the following iteration scheme: $y^{(0)} \equiv 1$ and $y^{(j+1)} = Ty^{(j)}$ for $j = 0, 1, 2, \ldots$. Clearly, in view of (3.8),

$$0 \le y_{m,n}^{(j+1)} \le y_{m,n}^{(j)} \le 1, \quad m \ge M,\ n \ge N,\ j \ge 0. \tag{3.11}$$

Thus as $j \to \infty$, $y^{(j)}$ converges pointwise to some nonnegative sequence $w = \{w_{m,n}\}$ which satisfies

$$x_{m,n}w_{m,n} = \frac{1}{\overline{p}}\left\{\sum_{(i,j)=(m,n+1)}^{\infty} x_{i+1,j}w_{i+1,j} + \sum_{(i,j)=(m,n)}^{\infty} \overline{q}_{i,j}\overline{f}\left(x_{i-k,j-l}w_{i-k,j-l}\right)\right\} + \frac{1-\overline{p}}{\overline{p}}\left\{\sum_{i=m}^{\infty} x_{i+1,n}w_{i+1,n} + \sum_{(i,j)=(m,n)}^{\infty} x_{i,j+1}w_{i,j+1}\right\} \tag{3.12}$$

for $m \geq M$ and $n \geq N$ and $w_{m,n} = 1$ elsewhere. Taking differences on both sides of the above equality, we see that the double sequence $\{u_{m,n}\} = \{x_{m,n}w_{m,n}\}$ is an eventually nonnegative solution of (3.4). Finally, we claim that $\{u_{m,n}\}$ is eventually positive, provided $\overline{q}_{m,n} > 0$ for $m \geq M$ and $n \geq N$. To see this, suppose to the contrary that there exists a pair of integers $m^* \geq M$ and $n^* \geq N$ such that $u_{m,n} > 0$ for $(m,n) \in \{M-k, M-k+1, \dots, m^*\} \times \{N-l, N-l+1, \dots, n^*\} \backslash \{(m^*, n^*)\}$ but $u_{m^*,n^*} = 0$. Then in view of (3.12),

$$0 \geq \sum_{(i,j)=(m^*,n^*+1)}^{\infty} u_{i+1,j} + \sum_{(i,j)=(m^*,n^*)}^{\infty} \overline{q}_{i,j}\overline{f}\left(u_{i-k,j-l}\right), \tag{3.13}$$

which implies $u_{i,j} = 0$ for $i \geq m^* + 1$ and $j \geq n^* + 1$, as well as

$$\overline{q}_{i,j}\overline{f}\left(u_{i-k,j-l}\right) = 0 \tag{3.14}$$

for $i \geq m^*$ and $j \geq n^*$. This contradicts our assumptions that $\overline{q}_{m^*,n^*} > 0$ and $u_{m^*-k,n^*-l} > 0$. The proof is complete. □

As an immediate consequence of Theorem 3.1, we have the following connection between the partial difference inequality (3.2) and the partial difference equation (3.3).

Corollary 3.2. *Suppose* $0 < p \leq 1$, $\{q_{m,n}\}$ *is eventually positive and* f *is positive and nondecreasing for* $x > 0$. *Then* (3.2) *has an eventually positive solution if and only if* (3.3) *has an eventually positive solution.*

In order to establish the desired connections between (3.1) and (3.3), we first recall a few facts for (3.1) from Section 2.2.

Every proper solution of (3.1) oscillates if and only if the following characteristic equation has no positive roots:

$$\lambda + \mu - p + q\lambda^{-k}\mu^{-l} = 0. \tag{3.15}$$

Next, note that when $p \in (0,1]$ and $q \geq 0$, every eventually positive solution of (3.1) or (3.3) is proper. Indeed, if $x = \{x_{m,n}\}$ is such a solution, then

$$x_{m,n+1} + x_{m+1,n} - px_{m,n} \leq 0 \tag{3.16}$$

eventually, so that x is eventually decreasing in m and also in n. As a consequence, when $p \in (0,1]$ and $q \geq 0$, every solution of (3.1) is oscillatory if and only if every proper solution oscillates.

Next, note that when $q > 0$, inequality (2.22) will still be valid when q is decreased and p is increased by sufficiently small perturbations. Thus the following continuous dependence of parameters theorem for (3.1) holds.

Theorem 3.3. *Suppose that $p, q > 0$ and that every proper solution of* (3.1) *is oscillatory. Then there exists a nonnegative number $\xi_1 > -p$ and a positive number $\xi_2 < q$ such that for every $\epsilon_1 \in [0, \xi_1]$ and $\epsilon_2 \in [0, \xi_2]$, each proper solution of the following equation is also oscillatory:*

$$x_{m+1,n} + x_{m,n+1} - (p + \epsilon_1)x_{m,n} + (q - \epsilon_2)x_{m-k,n-l} = 0, \quad (m,n) \in N_0^2. \tag{3.17}$$

We are ready to establish several important relations between the linear equation (3.1) and the nonlinear equation (3.3).

Theorem 3.4. *Suppose $p \in (0,1]$. Suppose further that*

$$\liminf_{m,n\to\infty} q_{m,n} \geq q > 0. \tag{3.18}$$

If there is an eventually positive sequence $u = \{u_{m,n}\}$ which satisfies

$$x_{m+1,n} + x_{m,n+1} - px_{m,n} + q_{m,n}x_{m-k,n-l} \leq 0 \tag{3.19}$$

for all large m and n, then (3.1) *has an eventually positive solution.*

Proof. In view of (3.18), for any $\epsilon \in (0,q)$, $q_{m,n} > q - \epsilon$ for all large m and n. If (3.19) has an eventually positive solution, then by Theorem 3.1, the equation

$$x_{m+1,n} + x_{m,n+1} - px_{m,n} + (q - \epsilon)x_{m-k,n-l} = 0 \tag{3.20}$$

also has an eventually positive solution. Therefore, if every solution of (3.1) is oscillatory, then by Theorem 3.3, there will exist an $\epsilon_0 \in (0,q)$ such that (every proper and hence) every solution of

$$x_{m+1,n} + x_{m,n+1} - px_{m,n} + (q - \epsilon_0)x_{m-k,n-l} = 0 \tag{3.21}$$

oscillates. This is the desired contradiction.

As an immediate application, suppose that $f(x) \geq x$ for $x > 0$ and that (3.18) holds. If (3.3) has an eventually positive solution $u = \{u_{m,n}\}$, then

$$\begin{gathered} 0 = u_{m,n+1} + u_{m+1,n} - pu_{m,n} + q_{m,n}\frac{f(u_{m-k,n-l})}{u_{m-k,n-l}}u_{m-k,n-l}, \\ \liminf_{m,n\to\infty} q_{m,n}\frac{f(u_{m-k,n-l})}{u_{m-k,n-l}} \geq \liminf_{m,n\to\infty} q_{m,n} \geq q \end{gathered} \tag{3.22}$$

would imply, by means of Theorem 3.4, that (3.1) will also have an eventually positive solution. □

Theorem 3.5. *Suppose that* $p \in (0,1]$, $f(x) \geq x$ *for* $x > 0$, *and* (3.18) *holds. If* (3.3) *has an eventually positive solution, then so does* (3.1).

Similar reasoning also leads to the following: suppose that $p \in (0,1]$, that (3.18) holds, and that

$$\liminf_{x\to 0^+} \frac{f(x)}{x} \geq 1. \tag{3.23}$$

If (3.3) has an eventually positive solution $x = \{x_{m,n}\}$ which satisfies $\lim_{m,n\to\infty} x_{m,n} = 0$, then (3.1) has an eventually positive solution.

It is not difficult to impose conditions such that all eventually positive solutions of (3.3) converge to zero as m, n tend to infinity. For example, assume that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_{m,n} = \infty. \tag{3.24}$$

In fact, for any eventually positive solution $x = \{x_{m,n}\}$ of (3.3) where $0 < p \leq 1$, since it is decreasing in m and n eventually, we may assume that x tends to a nonnegative constant $\bar{x}$. If $\bar{x} > 0$, then assuming $x_{m,n} > 0$ for $m \geq M - k$ and $n \geq N - l$, we see from (3.8) that

$$p x_{m,n} \geq \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} q_{i,j} f(x_{i-k,j-l}). \tag{3.25}$$

Assuming f is continuous or nondecreasing on $(0,\infty)$, the infinite series of the above inequality will diverge to positive infinity, which is a contradiction. This shows that $\bar{x} = 0$. Finally, note that the condition (3.24) follows from (3.18). The following result is now clear.

Theorem 3.6. *Suppose that* $p \in (0,1]$, (2.8) *and* (2.11) *hold and* f *is either continuous or nondecreasing on* $(0,\infty)$. *If* (3.3) *has an eventually positive solution, then so does* (3.1).

We now turn to the question as to when the existence of an eventually positive solution of (3.1) implies the existence of eventually positive solutions of (3.3).

Theorem 3.7. *Suppose that* $p \in (0,1]$, $0 < q_{m,n} \leq q$ *for all large* m *and* n, *and* $f(x) \leq x$ *for all* x *in a nonempty right neighborhood* $(0,\delta)$ *of zero. If* (3.1) *has an eventually positive solution, then so does* (3.3).

Proof. Suppose (3.1) has an eventually positive solution. Then by Theorem 2.1, the characteristic equation will be satisfied by a pair of positive numbers λ_0 and μ_0.

It is not difficult to check that the sequence $\{x_{m,n}\}$ defined by $\{\lambda_0^m \mu_0^n\}$ is an eventually positive solution of (3.1). Furthermore, since it is easily seen from the characteristic equation that $\lambda_0 + \mu_0 < p \le 1$, we see that $x_{m,n} \to 0$ as m, n tend to infinity. Therefore, $f(x_{m,n}) \le x_{m,n}$ for all large m and n. As a consequence,

$$\begin{aligned} &x_{m+1,n} + x_{m,n+1} - px_{m,n} + q_{m,n} f(x_{m-k,n-l}) \\ &\quad \le x_{m+1,n} + x_{m,n+1} - px_{m,n} + qx_{m-k,n-l} = 0 \end{aligned} \tag{3.26}$$

for all large m and n. We now see from Theorem 3.1 that (3.3) will have an eventually positive solution. The proof is complete. □

Now it is a position to state a linearized oscillation theorem.

In the oscillation theory, it is desirable to show that a nonlinear equation, when appropriate conditions are imposed, has the same oscillatory behavior as an associated linear equation. The following result follows directly from Theorems 3.6 and 3.7.

Theorem 3.8. *Suppose that* $p \in (0, 1]$, $q > 0$, k, l *are nonnegative integers such that* $\min(k, l) > 0$, *and* $f : R \to R$ *is either continuous or nondecreasing on* $(0, \infty)$. *Suppose further that* $0 < f(x) \le x$ *for all* x *in a (nonempty) right neighborhood* $(0, \delta)$ *of zero and that* $\liminf_{x\to 0^+}(f(x)/x) = 1$. *Then*

$$x_{m+1,n} + x_{m,n+1} - px_{m,n} + qx_{m-k,n-l} = 0, \quad m, n = 0, 1, 2, \dots, \tag{3.27}$$

has an eventually positive solution if and only if

$$x_{m+1,n} + x_{m,n+1} - px_{m,n} + qf(x_{m-k,n-l}) = 0, \quad m, n = 0, 1, 2, \dots, \tag{3.28}$$

has an eventually positive solution.

Each of the previous results related to (3.2) and (3.3) has a dual statement valid for eventually negative solutions. This is clear from the fact that $\{x_{m,n}\}$ is a solution of (3.3) if and only if $\{-x_{m,n}\}$ is a solution of

$$y_{m+1,n} + y_{m,n+1} - py_{m,n} + q_{m,n}F(y_{m-k,n-l}) = 0, \quad m, n = 0, 1, 2, \dots, \tag{3.29}$$

where $F(t) = -f(-t)$ for $t \in R$. Note that $\operatorname{sgn} F(t) = \operatorname{sgn} t$ for $t \ne 0$, and F is nondecreasing on $(0, \infty)$ when f is nondecreasing on $(0, \infty)$. Thus, if in the above theorem, we assume several additional dual conditions, then we may conclude that every solution of (3.27) oscillates if and only if every solution of (3.28) oscillates.

Theorem 3.9. *Suppose that* $p \in (0, 1]$, $q > 0$, k, l *are nonnegative integers such that* $\min(k, l) > 0$ *and* $f : R \to R$ *is either continuous or nondecreasing on* $(-\infty, \infty)\backslash\{0\}$. *Suppose further that* $xf(x) > 0$ *for all* $x \ne 0$ *and* $0 < f(x)/x \le 1$ *in a (nonempty) deleted neighborhood* $(-\delta, \delta)\backslash\{0\}$ *and that* $\liminf_{x\to 0}(f(x)/x) = 1$. *Then every solution of* (3.27) *oscillates if and only if every solution of* (3.28) *oscillates.*

Example 3.10. Consider the partial difference equation

$$x_{m+1,n}+x_{m,n+1}-px_{m,n}+q\frac{x_{m-k,n-l}}{1+x_{m-k,n-l}^2}=0,\quad m,n=0,1,2,\ldots, \tag{3.30}$$

where $p\in(0,1]$, $q>0$ and k, l are nonnegative integers. By Theorem 3.8 and its following remarks, we see that every solution of this equation oscillates if and only if every solution of (3.27) oscillates. In view of Theorem 2.3, we see further that every solution of this equation oscillates if and only if $q(k+l+1)^{k+l+1}>k^kl^lp^{k+l+1}$.

The linearized oscillation theorem for the delay partial difference equation

$$aA_{m+1,n+1}+bA_{m,n+1}-pA_{m,n}+q_{m,n}f(A_{m-k,n-l})=0,\quad (m,n)\in N_0^2 \tag{3.31}$$

has been also established.

3.2.2. Linearized oscillation for $A_{m-1,n}+A_{m,n-1}-pA_{m,n}+q_{m,n}f(x_{m+k,n+l})=0$

We consider a nonlinear advanced partial difference equation

$$A_{m-1,n}+A_{m,n-1}-pA_{m,n}+q_{m,n}f(A_{m+k,n+l})=0,\quad m,n=0,1,\ldots, \tag{3.32}$$

where $f\in C(R,R)$, $q_{m,n}\geq 0$ on N_0^2 and $k,l\in N_1$.

In this section, we will show some linearized oscillation theorems for (3.32). Next, we will show an existence result for positive solutions of (3.32). Finally, we will obtain a comparison theorem.

Consider (3.32) together with the linear equation

$$A_{m-1,n}+A_{m,n-1}-pA_{m,n}+qA_{m+k,n+l}=0, \tag{3.33}$$

where k and l are positive integers and $p,q>0$.

From Chapter 2, we have the following result.

Lemma 3.11. *The following statements are equivalent.*

(a) *Every proper solution of* (3.33) *oscillates.*

(b) *The characteristic equation*

$$\lambda^{-1}+\mu^{-1}-p+q\lambda^k\mu^l=0 \tag{3.34}$$

has no positive roots.

(c)

$$q\frac{(k+l+1)^{k+l+1}}{k^kl^lp^{k+l+1}}>1, \tag{3.35}$$

where $0^0=1$.

(λ,μ) is said to be a positive root of (3.34) if it satisfies (3.34) and $\lambda > 0$, $\mu > 0$. Similar to Theorem 3.3, from Lemma 3.11, we can obtain the following result easily.

Lemma 3.12. *Assume that every proper solution of* (3.33) *oscillates. Then there exists* $\epsilon_0 \in (0,q)$ *such that for each* $\epsilon \in [0,\epsilon_0]$, *every proper solution of the equation*

$$A_{m-1,n} + A_{m,n-1} - pA_{m,n} + (q-\epsilon)A_{m+k,n+l} = 0 \tag{3.36}$$

also oscillates.

Theorem 3.13. *Assume that*

(i) $\liminf_{m,n\to\infty} q_{m,n} = q > 0$, $p \in (0,1]$,

(ii) $f(x)/x > 0$ *for* $|x| \geq c > 0$ *and* $\lim_{x\to\infty}(f(x)/x) = 1$.

Then every proper solution of (3.33) *oscillates implies that every proper solution of* (3.32) *oscillates.*

Proof. Suppose to the contrary that $\{A_{m,n}\}$ is an eventually positive proper solution of (3.32). Then there exist m_0 and n_0 such that $A_{m,n} > 0$ for $m \geq m_0$, $n \geq n_0$. Hence $A_{m-1,n} < A_{m,n}$ and $A_{m,n-1} < A_{m,n}$, that is, $A_{m,n}$ is increasing in m and n. If $\lim_{m,n\to\infty} A_{m,n} = L > 0$, L is finite. From (3.32), we have $(2-p)L+qf(L) \leq 0$, which is a contradiction. Therefore $\lim_{m,n\to\infty} A_{m,n} = \infty$. Similarly, we have $\lim_{n\to\infty} A_{m,n} = \infty$ and $\lim_{m\to\infty} A_{m,n} = \infty$. Let

$$\overline{q}_{m,n} = q_{m,n}\frac{f(A_{m+k,n+l})}{A_{m+k,n+l}}. \tag{3.37}$$

Then $\liminf_{m,n\to\infty} \overline{q}_{m,n} = q$. For each $\epsilon \in (0,\epsilon_0]$ there exist $M > m_0$ and $N > n_0$ such that $\overline{q}_{m,n} > q-\epsilon$, for $m \geq M-1$, $n \geq N-1$. Therefore

$$A_{m-1,n} + A_{m,n-1} - pA_{m,n} + (q-\epsilon)A_{m+k,n+l} \leq 0 \tag{3.38}$$

for $m \geq M-1$, $n \geq N-1$.

Summing (3.38) in n from N to n, we have

$$\sum_{i=N}^{n} A_{m-1,i} + (1-p)\sum_{i=N}^{n} A_{m,i-1} + p\sum_{i=N}^{n}\left(A_{m,i-1} - A_{m,i}\right) + (q-\epsilon)\sum_{i=N}^{n} A_{m+k,i+l} \leq 0. \tag{3.39}$$

Hence

$$\sum_{i=N}^{n} A_{m-1,i} + (1-p)\sum_{i=N}^{n} A_{m,i-1} + pA_{m,N-1} - pA_{m,n} + (q-\epsilon)\sum_{i=N}^{n} A_{m+k,i+l} \leq 0. \tag{3.40}$$

We rewrite the above inequality in the form

$$A_{m,N-1} + A_{m-1,n} - A_{m,n} + \sum_{i=N}^{n-1} A_{m-1,i} + (1-p)\sum_{i=N+1}^{n+1} A_{m,i-1} + (q-\epsilon)\sum_{i=N}^{n} A_{m+k,i+l} \le 0. \tag{3.41}$$

Summing (3.41) in m from M to m, we get

$$-A_{m,n} + A_{M-1,n} + \sum_{j=M}^{m}\sum_{i=N}^{n-1} A_{j-1,i} + (1-p)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} A_{j,i-1} + \sum_{j=M}^{m} A_{j,N-1} + (q-\epsilon)\sum_{j=M}^{m}\sum_{i=N}^{n} A_{j+k,i+l} \le 0. \tag{3.42}$$

Thus

$$A_{m,n} \ge \sum_{j=M}^{m}\sum_{i=N}^{n-1} A_{j-1,i} + A_{M-1,n} + (1-p)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} A_{j,i-1} + \sum_{j=M}^{m} A_{j,N-1} + (q-\epsilon)\sum_{j=M}^{m}\sum_{i=N}^{n} A_{j+k,i+l}, \quad m \ge M,\ n \ge N. \tag{3.43}$$

Define the set of real double sequences

$$X = \left\{\{B_{m,n}\} \mid 0 \le B_{m,n} \le 1,\ m \ge M-1,\ n \ge N-1\right\} \tag{3.44}$$

and an operator T on X by

$$(TB)_{m,n} = \begin{cases} \dfrac{1}{A_{m,n}}\Bigg[\displaystyle\sum_{j=M}^{m}\sum_{i=N}^{n-1} A_{j-1,i}B_{j-1,i} + A_{M-1,n}B_{M-1,n} \\ \quad +(1-p)\displaystyle\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} A_{j,i-1}B_{j,i-1} \\ \quad +\displaystyle\sum_{j=M}^{m} A_{j,N-1}B_{j,N-1} \\ \quad +(q-\epsilon)\displaystyle\sum_{j=M}^{m}\sum_{i=N}^{n} A_{j+k,i+l}B_{j+k,i+l}\Bigg], & m \ge M,\ n \ge N, \\ 1, & \text{otherwise}. \end{cases} \tag{3.45}$$

In view of (3.43), we see that $TX \subset X$. Define $\{B_{m,n}^{(i)}\}$, $i = 0, 1, \ldots$, as follows:

$$B_{m,n}^{(0)} \equiv 1, \qquad B_{m,n}^{(r)} = (TB)_{m,n}^{(r-1)}, \quad r = 1, 2, \ldots. \tag{3.46}$$

By induction and (3.43), we can prove that

$$B_{m,n}^{(0)} \geq B_{m,n}^{(1)} \geq \cdots \geq B_{m,n}^{(r)} \geq B_{m,n}^{(r+1)} \geq \cdots \tag{3.47}$$

for $m \geq M-1$, $n \geq N-1$. Thus the limit $B_{m,n} = \lim_{r\to\infty} B_{m,n}^{(r)}$ exists and

$$B_{m,n} = \begin{cases} \dfrac{1}{A_{m,n}}\Bigg[\sum_{j=M}^{m}\sum_{i=N}^{n-1} A_{j-1,i}B_{j-1,i} + A_{M-1,n}B_{M-1,n} \\ \quad +(1-p)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} A_{j,i-1}B_{j,i-1} \\ \quad +\sum_{j=M}^{m} A_{j,N-1}B_{j,N-1} \\ \quad +(q-\epsilon)\sum_{j=M}^{m}\sum_{i=N}^{n} A_{j+k,i+l}B_{j+k,i+l}\Bigg], & m \geq M,\ n \geq N, \\ 1, & \text{otherwise.} \end{cases} \tag{3.48}$$

Clearly, $B_{m,n} > 0$ for $m \geq M-1$, $n \geq N-1$. Set $x_{m,n} = A_{m,n}B_{m,n}$. Then $x_{m,n} > 0$, $m \geq M-1$, $n \geq N-1$, and

$$\begin{aligned} x_{m,n} = {} & \sum_{j=M}^{m}\sum_{i=N}^{n-1} x_{j-1,i} + x_{M-1,n} + (1-p)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} x_{j,i-1} \\ & + \sum_{j=M}^{m} x_{j,N-1} + (q-\epsilon)\sum_{j=M}^{m}\sum_{i=N}^{n} x_{j+k,i+l}, \quad m \geq M,\ n \geq N. \end{aligned} \tag{3.49}$$

From the last equation, we get

$$x_{m-1,n} - x_{m,n} = -\sum_{i=N}^{n-1} x_{m-1,i} - (1-p)\sum_{i=N+1}^{n+1} x_{m,i-1} - x_{m,N-1} - (q-\epsilon)\sum_{i=N}^{n} x_{m+k,i+l}, \tag{3.50}$$

or

$$x_{m,n} = \sum_{i=N}^{n} x_{m-1,i} + (1-p)\sum_{i=N+1}^{n+1} x_{m,i-1} + x_{m,N-1} + (q-\epsilon)\sum_{i=N}^{n} x_{m+k,i+l}. \tag{3.51}$$

Hence

$$\begin{aligned} x_{m,n-1} - x_{m,n} &= \sum_{i=N}^{n-1} x_{m-1,i} + (1-p)\sum_{i=N+1}^{n} x_{m,i-1} + (q-\epsilon)\sum_{i=N}^{n-1} x_{m+k,i+l} \\ &\quad - \sum_{i=N}^{n} x_{m-1,i} - (1-p)\sum_{i=N+1}^{n+1} x_{m,i-1} - (q-\epsilon)\sum_{i=N}^{n} x_{m+k,i+l} \\ &= -x_{m-1,n} - (1-p)x_{m,n} - (q-\epsilon)x_{m+k,n+l}, \end{aligned} \tag{3.52}$$

that is, (3.36) has a positive solution $\{x_{m,n}\}$. In view of $x_{i,j} \le A_{i,j}$ for all large i and j, $\{x_{i,j}\}$ is a proper solution. By Lemma 3.12, (3.33) has a positive proper solution, which is a contradiction. The proof is complete. □

Theorem 3.14. *Assume that*

(i) $0 \le p_{m,n} \le p$,

(ii) *there exists a positive number c such that $f(x)$ is nondecreasing in x for $|x| \ge c$ and*

$$0 \le \frac{f(x)}{x} \le 1, \quad |x| \ge c. \tag{3.53}$$

Suppose (3.33) *has a positive proper solution, then* (3.32) *also has a positive proper solution.*

Proof. If (3.33) has a positive proper solution, by Lemma 3.11, its characteristic equation (3.34) has a positive root (λ, μ) with $\lambda^{-1} + \mu^{-1} < p$ and $\{A_{m,n}\} = \{\lambda^m \mu^n\}$ is a positive solution of (3.33).

Since $\lambda > 1$ and $\mu > 1$, this is an unbounded solution. There exists $c > 0$ such that

$$A_{m,n} \ge c, \quad m \ge M-1,\ n \ge N-1. \tag{3.54}$$

In view of condition (ii), $f(A_{m,n}) \le A_{m,n}$. Similar to the proof of Theorem 3.13, summing (3.33) we can get

$$\begin{aligned} A_{m,n} &= \sum_{j=M}^{m}\sum_{i=N}^{n-1} A_{j-1,i} + A_{M-1,n} + (1-p)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} A_{j,i-1} \\ &\quad + \sum_{j=M}^{m} A_{j,N-1} + q\sum_{j=M}^{m}\sum_{i=N}^{n} A_{j+k,i+l}, \end{aligned} \tag{3.55}$$

and hence

$$A_{m,n} \geq \sum_{j=M}^{m}\sum_{i=N}^{n-1} A_{j-1,i} + A_{M-1,n} + (1-p)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} A_{j,i-1} + \sum_{j=M}^{m} A_{j,N-1} + \sum_{j=M}^{m}\sum_{i=N}^{n} q_{j,i} f\left(A_{j+k,i+l}\right). \tag{3.56}$$

Similar to the proof of Theorem 3.13, we can prove that the equation

$$x_{m,n} = \sum_{j=M}^{m}\sum_{i=N}^{n-1} x_{j-1,i} + x_{M-1,n} + (1-p)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} x_{j,i-1} + \sum_{j=M}^{m} x_{j,N-1} + \sum_{j=M}^{m}\sum_{i=N}^{n} q_{j,i} f\left(x_{j+k,i+l}\right) \tag{3.57}$$

has a positive solution $\{x_{m,n}\}$ with $x_{m,n} \leq A_{m,n}$, which implies that $\{x_{m,n}\}$ is a positive proper solution of (3.32). The proof is complete. □

Combining Theorems 3.13 and 3.14 we obtain the following result.

Theorem 3.15. *Assume that* $q_{m,n} \equiv q > 0$, *(ii) of Theorem 3.13 and (ii) of Theorem 3.14 hold. Then every proper solution of* (3.32) *oscillates if and only if every proper solution of* (3.33) *oscillates.*

Corollary 3.16. *Assume that (ii) of Theorem 3.14 holds and*

$$0 \leq q_{m,n} \leq \frac{k^k l^l p^{k+l+1}}{(k+l+1)^{k+l+1}}. \tag{3.58}$$

Then (3.32) *has a positive solution.*

Example 3.17. Consider the nonlinear partial difference equation

$$A_{m-1,n} + A_{m,n-1} - pA_{m,n} + q\frac{A^3_{m+k,n+l}}{1 + A^2_{m+k,n+l}} = 0, \tag{3.59}$$

where $p \in (0,1]$, $q > 0$, $k, l \in N_0$. By Theorem 3.15, every solution of this equation is oscillatory if and only if

$$q > \frac{k^k l^l p^{k+l+1}}{(k+l+1)^{k+l+1}}. \tag{3.60}$$

Now we compare the equation

$$A_{m-1,n} + A_{m,n-1} - pA_{m,n} + q_{m,n}A_{m+k,n+l} = 0 \tag{3.61}$$

and the equation

$$A_{m-1,n} + A_{m,n-1} - rA_{m,n} + s_{m,n}A_{m+k,n+l} = 0. \tag{3.62}$$

Theorem 3.18. *Assume that* $0 < p \leq r \leq 1$ *and* $q_{m,n} \geq s_{m,n}$ *for all large* m *and* n. *Then every solution of* (3.62) *is oscillatory implies the same for* (3.61).

Proof. Suppose to the contrary, let $\{A_{m,n}\}$ be a positive solution of (3.61). As before, by summing (3.61), we can derive

$$\begin{aligned} A_{m,n} &= \sum_{j=M}^{m}\sum_{i=N}^{n-1} A_{j-1,i} + A_{M-1,n} + (1-p)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} A_{j,i-1} \\ &\quad + \sum_{j=M}^{m} A_{j,N-1} + \sum_{j=M}^{m}\sum_{i=N}^{n} q_{j,i}A_{j+k,i+l} \\ &\geq \sum_{j=M}^{m}\sum_{i=N}^{n-1} A_{j-1,i} + A_{M-1,n} + (1-r)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} A_{j,i-1} \\ &\quad + \sum_{j=M}^{m} A_{j,N-1} + \sum_{j=M}^{m}\sum_{i=N}^{n} s_{j,i}A_{j+k,i+l}. \end{aligned} \tag{3.63}$$

As before, the last inequality implies that the equation

$$\begin{aligned} x_{m,n} &= \sum_{j=M}^{m}\sum_{i=N}^{n-1} x_{j-1,i} + x_{M-1,n} + (1-r)\sum_{j=M}^{m}\sum_{i=N+1}^{n+1} x_{j,i-1} \\ &\quad + \sum_{j=M}^{m} x_{j,N-1} + \sum_{j=M}^{m}\sum_{i=N}^{n} s_{j,i}x_{j+k,i+l} \end{aligned} \tag{3.64}$$

has a positive solution, and hence (3.62) has a positive solution, this contradiction proves the theorem. □

Remark 3.19. The above results can be extended to the more general equation

$$A_{m-1,n} + A_{m,n-1} - pA_{m,n} + \sum_{i=1}^{u} q_i(m,n)f_i\left(A_{m+k_i,n+l_i}\right) = 0. \tag{3.65}$$

3.2.3. Linearized oscillation for the equation with continuous arguments

In this section, we attempt to show the linearized oscillation theorems for the nonlinear partial difference equation with continuous arguments of the form

$$A(x+1,y) + A(x,y+1) - A(x,y) + p(x,y)f\left(A(x-\sigma,y-\tau)\right) = 0, \tag{3.66}$$

where $x \geq 0$, $y \geq 0$, $f \in C(R,R)$, $uf(u) > 0$ for $u \neq 0$, $p(x,y) > 0$, $\tau, \sigma > 0$.

Consider (3.66) together with the linear equation

$$A(x+1,y)+A(x,y+1)-A(x,y)+pA(x-\sigma,y-\tau)=0, \tag{3.67}$$

where $p,\tau,\sigma>0$.

From Section 2.4, we have the following lemma.

Lemma 3.20. *The following statements are equivalent.*

(a) *Every solution of* (3.67) *oscillates.*

(b) *The characteristic equation*

$$\lambda+\mu-1+p\lambda^{-\sigma}\mu^{-\tau}=0 \tag{3.68}$$

has no positive roots.

(c)

$$p>\frac{\sigma^{\sigma}\tau^{\tau}}{(\sigma+\tau+1)^{\sigma+\tau+1}}. \tag{3.69}$$

From Lemma 3.20, we obtain the following lemma.

Lemma 3.21. *Assume that every proper solution of* (3.67) *oscillates, then there exists* $\epsilon_0\in(0,p)$ *such that for each* $\epsilon\in[0,\epsilon_0]$, *every proper solution of the equation*

$$A(x+1,y)+A(x,y+1)-A(x,y)+(p-\epsilon)A(x-\sigma,y-\tau)=0 \tag{3.70}$$

also oscillates.

Lemma 3.22. *Let* $A(x,y)$ *be an eventually positive solution of* (3.67) *and*

$$Z(x,y)=\int_x^{x+1}\int_y^{y+1}A(s,q)\,ds\,dq. \tag{3.71}$$

Then $\partial Z/\partial x<0$, $\partial Z/\partial y<0$, *and* $\lim_{x,y\to+\infty}Z(x,y)=0$.

Proof. There exist sufficiently large x_0 and y_0 such that $A(x,y)>0$ for $x\geq x_0$, $y\geq y_0$. From (3.67), we have $A(x,y)>A(x+1,y)$, $A(x,y)>A(x,y+1)$ for $x\geq x_0+\sigma$, $y\geq y_0+\tau$. By (3.71), we obtain

$$\begin{aligned}\frac{\partial Z}{\partial x}&=\int_y^{y+1}\left[A(x+1,q)-A(x,q)\right]dq<0,\\ \frac{\partial Z}{\partial y}&=\int_x^{x+1}\left[A(s,y+1)-A(s,y)\right]ds<0.\end{aligned} \tag{3.72}$$

Since $Z(x, y) > 0$, the limit $\lim_{x,y\to+\infty} Z(x, y) = d$ exists. We claim that $d = 0$. Otherwise, $d > 0$, integrating (3.67) in y from y to $y + 1$ and in x from x to $x + 1$, we have

$$Z(x+1, y) + Z(x, y+1) - Z(x, y) + pZ(x-\sigma, y-\tau) = 0. \tag{3.73}$$

Taking the limit on both sides of (3.73), we obtain $d + pd = 0$, which is a contradiction. Therefore we have $\lim_{x,y\to+\infty} Z(x, y) = 0$. The proof is complete. □

Lemma 3.23 (Jensen integral inequality). *If $\phi(u)$ is a continuous convex function, $f(x)$ and $p(x)$ are continuous functions on $[a, b]$, $p(x) \ge 0$, $\int_a^b p(x)dx > 0$, then the following inequality holds:*

$$\phi\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}\right) \le \frac{\int_a^b p(x)\phi(f(x))dx}{\int_a^b p(x)dx}. \tag{3.74}$$

Lemma 3.24. *Assume that $f(u)$ is nondecreasing and is convex as $u \ge 0$. Set $p(x, y) = \min\{p(s, q) : x \le s \le x + 1,\ y \le q \le y + 1\}$. Then (3.66) has an eventually positive solution if and only if the inequality*

$$A(x+1, y) + A(x, y+1) - A(x, y) + p(x, y)f\big(A(x-\sigma, y-\tau)\big) \le 0 \tag{3.75}$$

has an eventually positive solution.

Proof. The necessity is obvious, we only need to prove the sufficiency. Let $A(x, y)$ be an eventually positive solution of (3.75) and let $Z(x, y)$ be defined by (3.71), so there exist sufficiently large x_0 and y_0 such that $A(x, y) > 0$ for $x \ge x_0$, $y \ge y_0$. From (3.75), we have

$$Z(x+1, y) + Z(x, y+1) - Z(x, y) + \int_x^{x+1}\int_y^{y+1} p(s, q)f\big(x(s-\sigma, q-\tau)\big)dq\,ds \le 0. \tag{3.76}$$

By Lemma 3.23, we have

$$Z(x+1, y+i) + Z(x, y+1+i) - Z(x, y+i) + p(x, y+i)f\big(Z(x-\sigma, y-\tau+i)\big) \le 0 \tag{3.77}$$

for $x \ge x_0$, $y \ge y_0$, $i \in N_0$. Summing (3.77) in i from 0 to $+\infty$, we get

$$\sum_{i=0}^{+\infty} Z(x+1, y+i) - Z(x, y) + \sum_{i=0}^{+\infty} p(x, y+i)f\big(Z(x-\sigma, y+i-\tau)\big) \le 0. \tag{3.78}$$

From the above inequality, we have

$$Z(x+1+j,y)-Z(x+j,y)+\sum_{i=1}^{+\infty}Z(x+1+j,y+i)$$
$$+\sum_{i=0}^{+\infty}p(x+j,y+i)f\big(Z(x-\sigma+j,y+i-\tau)\big)\leq 0, \tag{3.79}$$

where $j\in N_0$.

Summing (3.79) in j from 0 to $+\infty$, we obtain

$$Z(x,y)\geq\sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty}Z(x+j,y+i)+\sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty}p(x+j,y+i)f\big(Z(x-\sigma+j,y+i-\tau)\big). \tag{3.80}$$

Define a set of continuous functions

$$X=\{B(x,y)\in C\mid 0\leq B(x,y)\leq Z(x,y),\ x\geq x_0-\sigma,\ y\geq y_0-\tau\} \tag{3.81}$$

and an operator T on X by

$$TB(x,y)=\begin{cases}\displaystyle\sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty}B(x+j,y+i)+\sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty}p(x+j,y+i) & \\ \qquad\times f\big(B(x-\sigma+j,y-\tau+i)\big), & x\geq x_0,\ y\geq y_0,\\ TB(x_0,y_0)+Z(x,y)-Z(x_0,y_0), & \text{otherwise.}\end{cases} \tag{3.82}$$

In view of (3.80), we see that $TX\subset X$.

Define $B^{(i)}(x,y)$, $i=0,1,2,\ldots$, as follows:

$$B^{(0)}(x,y)=Z(x,y),\qquad B^{(n)}(x,y)=TB^{(n-1)}(x,y),\quad n=1,2,\ldots, \tag{3.83}$$

so $B^{(1)}(x,y)=TB^{(0)}(x,y)\leq B^{(0)}(x,y),\ldots$, by induction we can prove that

$$B^{(0)}(x,y)\geq B^{(1)}(x,y)\geq\cdots\geq B^{(n)}(x,y)\cdots\geq 0 \tag{3.84}$$

for $x\geq x_0-\sigma$, $y\geq y_0-\tau$. Then the limit $\lim_{n\to+\infty}B^{(n)}(x,y)=B(x,y)$ exists. Hence we have

$$B(x,y)=\sum_{j=1}^{+\infty}\sum_{i=1}^{+\infty}B(x+j,y+i)+\sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty}p(x+j,y+i)f\big(B(x-\sigma+j,y-\tau+i)\big) \tag{3.85}$$

for $x\geq x_0$, $y\geq y_0$. Clearly, $B(x,y)>0$ for $x\geq x_0$, $y\geq y_0$ and satisfies (3.66). The proof is complete. □

Theorem 3.25. *Assume that*

(i) $\liminf_{x,y\to+\infty} p(x,y) = p > 0$,

(ii) $f(z) \in C(R,R)$, $f(z)$ *is convex as* $z \geq 0$ *and is concave as* $z < 0$, $zf(z) > 0$, *for* $z \neq 0$, *and* $\lim_{z\to 0}(f(z)/z) = 1$.

Then every proper solution of (3.67) *oscillates implies that every solution of* (3.66) *oscillates.*

Proof. Suppose to the contrary, let $A(x,y)$ be an eventually positive solution of (3.66) and let $Z(x,y)$ be defined by (3.71). Then there exist sufficiently large x_0 and y_0 such that $A(x,y) > 0$ for $x \geq x_0$, $y \geq y_0$. From (3.66), we have

$$Z(x+1,y) + Z(x,y+1) - Z(x,y) + \int_x^{x+1}\int_y^{y+1} p(s,q)f(A(s-\sigma,q-\tau))\,dq\,ds = 0. \tag{3.86}$$

By condition (i), we see that for each $\epsilon_1 \in (0,\epsilon_0]$, there exist $X_1 > x_0$ and $Y_1 > y_0$ such that $p(x,y) \geq p - \epsilon_1$ for $x \geq X_1$, $y \geq Y_1$. So we have

$$Z(x+1,y)+Z(x,y+1)-Z(x,y)+(p-\epsilon_1)\int_x^{x+1}\int_y^{y+1} f(A(s-\sigma,q-\tau))\,dq\,ds \leq 0. \tag{3.87}$$

Then, by Lemma 3.23, we have

$$Z(x+1,y) + Z(x,y+1) - Z(x,y) + (p-\epsilon_1)f(Z(x-\sigma,y-\tau)) \leq 0. \tag{3.88}$$

By Lemma 3.22, $\lim_{x,y\to\infty} Z(x,y) = 0$ monotonically. Let

$$\overline{p}(x,y) = (p-\epsilon_1)\frac{f(Z(x-\sigma,y-\tau))}{Z(x-\sigma,y-\tau)}, \tag{3.89}$$

then

$$\lim_{x,y\to\infty} \overline{p}(x,y) = p - \epsilon_1. \tag{3.90}$$

So for each $\epsilon_2 \in [0,\epsilon_0]$ such that $\epsilon_1+\epsilon_2 \leq \epsilon_0$, there exist $X_2 > x_0$ and $Y_2 > y_0$ such that $\overline{p}(x,y) \geq p - \epsilon_1 - \epsilon_2$ for $x \geq X_2$, $y \geq Y_2$. Let $\epsilon = \epsilon_1 + \epsilon_2$, $X = \max\{X_1, X_2\}$, $Y = \max\{Y_1, Y_2\}$, then we have $\overline{p}(x,y) \geq p - \epsilon$ for $x \geq X$, $y \geq Y$. Therefore, from (3.88), we have

$$Z(x+1,y) + Z(x,y+1) - Z(x,y) + (p-\epsilon)Z(x-\sigma,y-\tau) \leq 0 \tag{3.91}$$

for $x \geq X$, $y \geq Y$, that is, the inequality

$$A(x+1,y) + A(x,y+1) - A(x,y) + (p-\epsilon)A(x-\sigma,y-\tau) \leq 0 \tag{3.92}$$

has an eventually positive solution $Z(x, y)$. So by Lemma 3.24, we obtain that (3.70) has a positive proper solution, which is a contradiction. The proof in the case of $A(x, y) < 0$ is similar. The proof is complete. □

Theorem 3.26. *Assume that*

(i) $0 \le p(x, y) \le P$, $\min\{p(s,q) : x \le s \le x+1,\ y \le q \le y+1\} = p(x, y)$,

(ii) *there exists a positive number α such that $f(z)$ is convex as $z \ge 0$ and is nondecreasing in $z \in [-\alpha, \alpha]$, $0 \le f(z)/z \le 1$ for $0 \le |z| < \alpha$.*

Suppose (3.67) *has a positive proper solution, then* (3.66) *has a positive solution.*

Proof. If (3.67) has a positive proper solution, by Lemma 3.20, its characteristic equation (3.68) has a positive root (λ, μ) with $0 < \lambda$, $\mu < 1$ and $\lambda^x \mu^y$ is a positive proper solution of (3.67). Choose $\delta > 0$ such that $A(x, y) = \delta\lambda^x\mu^y < \alpha$ for all $x \ge -\sigma$, $y \ge -\tau$. Obviously, $A(x, y)$ is a positive proper solution of (3.67) and satisfies $f(A(x, y)) \le A(x, y)$, by condition (i) and (3.67) we obtain that

$$A(x+1, y) + A(x, y+1) - A(x, y) + p(x, y) f\big(A(x-\sigma, y-\tau)\big) \le 0 \qquad (3.93)$$

has an eventually positive proper solution. By Lemma 3.24, we can obtain that (3.66) has an eventually positive solution. The proof is complete. □

Combining Lemma 3.24 and Theorem 3.25, we obtain the following result.

Corollary 3.27. *Assume that $p(x, y) \equiv p > 0$, (ii) of Theorem 3.25 holds, there exists a positive number α such that $f(z)$ is nondecreasing in $z \in [-\alpha, \alpha]$ and $0 \le f(z)/z \le 1$. Then every solution of* (3.66) *oscillates if and only if every solution of* (3.67) *oscillates.*

Corollary 3.28. *Assume that condition (ii) of Theorem 3.26 holds and*

$$0 \le p(x, y) \le \frac{\sigma^\sigma \tau^\tau}{(\sigma+\tau+1)^{\sigma+\tau+1}}, \qquad (3.94)$$

then (3.66) *has positive solutions.*

If $f(x) \equiv x$ and

$$\liminf_{x,y\to\infty} p(x, y) > \frac{\sigma^\sigma \tau^\tau}{(\sigma+\tau+1)^{\sigma+\tau+1}}, \qquad (3.95)$$

then every solution of (3.66) *oscillates.*

We consider (3.66) together with the equation

$$C(x+1, y) + C(x, y+1) - C(x, y) + q(x, y) g\big(C(x-\sigma, y-\tau)\big) = 0. \qquad (3.96)$$

We have a comparison result as follows.

Theorem 3.29. *Assume that* $p(x, y)$ *and* f *satisfy the assumptions in Lemma 3.24 and* $q(x, y) \geq p(x, y) \geq 0$, $ug(u) \geq uf(u)$, $u \neq 0$, *then every solution of* (3.66) *oscillates implies that every solution of* (3.96) *oscillates.*

Proof. Suppose to the contrary, let $C(x, y)$ be an eventually positive solution of (3.96). Then we have

$$\begin{aligned} &C(x+1, y) + C(x, y+1) - C(x, y) + p(x, y) f\big(C(x-\sigma, y-\tau)\big) \\ &\quad\leq C(x+1, y) + C(x, y+1) - C(x, y) + p(x, y) g\big(C(x-\sigma, y-\tau)\big) \\ &\quad\leq C(x+1, y) + C(x, y+1) - C(x, y) + q(x, y) g\big(C(x-\sigma, y-\tau)\big) = 0. \end{aligned} \tag{3.97}$$

Then by Lemma 3.24, we obtain that (3.66) has an eventually positive solution, which is a contradiction. The proof is complete. □

3.3. Nonlinear PDEs with variable coefficients

3.3.1. Oscillation for the equation (3.98)

Consider the equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} P_i(m, n) f_i\big(A_{m-k_i, n-l_i}\big) = 0, \tag{3.98}$$

where $P_i(m, n) \geq 0$, $k_i, l_i \in N_0$, $f_i \in C(R, R)$ and $x f_i(x) > 0$ for $x \neq 0$, $i = 1, 2, \ldots, u$.

Theorem 3.30. *Every solution of* (3.98) *is oscillatory provided that the following conditions hold:*

(i) *for* $1 \leq i \leq u$,

$$\liminf_{x \to 0} \frac{f_i(x)}{x} = S_i \in (0, \infty); \tag{3.99}$$

(ii) *for* $1 \leq i \leq u$,

$$\liminf_{m,n \to +\infty} P_i(m, n) = p_i > 0; \tag{3.100}$$

(iii) *for* $1 \leq i \leq u$, f_i *is nondecreasing;*

(iv)

$$\sum_{i=1}^{u} 2^{\eta_i} S_i p_i \frac{(\eta_i + 1)^{\eta_i + 1}}{\eta_i^{\eta_i}} > 1, \quad \text{where } \eta_i = \min\{k_i, l_i\}. \tag{3.101}$$

Proof. Suppose to the contrary that $\{A_{m,n}\}$ is an eventually positive solution of (3.98). By Lemma 2.62, $\lim_{m,n \to +\infty} A_{m,n} = \zeta \geq 0$.

We assert that $\zeta = 0$. Otherwise, if $\zeta > 0$, then by taking limits on both sides of (3.98), we have

$$0 \geq \zeta + \sum_{i=1}^{u} p_i f_i(\zeta) > 0, \tag{3.102}$$

which is a contradiction.

In view of (3.98), we have

$$\begin{aligned}
\frac{2A_{m+1,n+1}}{A_{m,n}} - 1 < \frac{A_{m,n+1} + A_{m+1,n}}{A_{m,n}} - 1 &= -\sum_{i=1}^{u} P_i(m,n)\frac{f_i(A_{m-k_i,n-l_i})}{A_{m,n}} \\
&< -\sum_{i=1}^{u} P_i(m,n)\frac{f_i(A_{m-\eta_i,n-\eta_i})}{A_{m,n}} \\
&= -\sum_{i=1}^{u} P_i(m,n)\frac{f_i(A_{m-\eta_i,n-\eta_i})}{A_{m-\eta_i,n-\eta_i}}\frac{A_{m-\eta_i,n-\eta_i}}{A_{m-\eta_i+1,n-\eta_i+1}}\cdots\frac{A_{m-1,n-1}}{A_{m,n}} \\
&= -\sum_{i=1}^{u} P_i(m,n)\frac{f_i(A_{m-\eta_i,n-\eta_i})}{A_{m-\eta_i,n-\eta_i}}\prod_{j=1}^{\eta_i}\frac{A_{m-j,n-j}}{A_{m-j+1,n-j+1}}
\end{aligned} \tag{3.103}$$

for all large m and n. Letting

$$\alpha_{m,n} = \frac{A_{m,n}}{A_{m+1,n+1}}, \tag{3.104}$$

we see that $\alpha_{m,n} > 1$ for all large m and n, and

$$\frac{2}{\alpha_{m,n}} + \sum_{i=1}^{u} P_i(m,n)\frac{f_i(A_{m-\eta_i,n-\eta_i})}{A_{m-\eta_i,n-\eta_i}}\prod_{j=1}^{\eta_i}\alpha_{m-j,n-j} < 1. \tag{3.105}$$

If $\{\alpha_{m,n}\}$ is unbounded, there exists a subsequence $\{\alpha_{m_s,n_t}\}$ such that

$$\limsup_{s,t\to+\infty} \alpha_{m_s-1,n_t-1} = +\infty. \tag{3.106}$$

But in view of the assumptions (i) and (ii), the left-hand side of (3.105) will not be bounded above. This contradiction shows that $\{\alpha_{m,n}\}$ is bounded above.

We now let $\xi = \liminf_{m,n\to+\infty} \alpha_{m,n}$. Then $\xi \in [1,+\infty)$. Furthermore, from (3.105), we see that

$$\frac{2}{\xi} \leq 1 - \sum_{i=1}^{u} p_i \xi^{\eta_i} S_i, \tag{3.107}$$

which implies $\xi > 2$ and

$$\sum_{i=1}^{u} p_i S_i \frac{\xi^{\eta_i+1}}{\xi - 2} \leq 1. \tag{3.108}$$

Note that

$$\min_{\xi>2} \frac{\xi^{\eta_i+1}}{\xi - 2} = 2^{\eta_i} \frac{(\eta_i + 1)^{\eta_i+1}}{\eta_i^{\eta_i}}, \tag{3.109}$$

thus we have

$$\sum_{i=1}^{u} p_i S_i 2^{\eta_i} \frac{(\eta_i + 1)^{\eta_i+1}}{\eta_i^{\eta_i}} \leq 1, \tag{3.110}$$

which is contrary to assumption (iv). The proof is complete. □

Theorem 3.31. *Every solution of* (3.98) *is oscillatory if conditions (i) and (iii) of Theorem 3.30 hold and for* $1 \leq t \leq u$,

$$\limsup_{m,n\to+\infty} \sum_{t=1}^{u} S_t \sum_{i=m}^{m+k_0} \sum_{j=n}^{n+l_0} P_t(i, j) > 1, \tag{3.111}$$

where $k_0 = \min\{k_1, k_2, \dots, k_u\}$ *and* $l_0 = \min\{l_1, l_2, \dots, l_u\}$.

Proof. Suppose to the contrary, let $\{A_{m,n}\}$ be an eventually positive solution of (3.98). Then, as in the proof of Theorem 3.30, we have $\lim_{m,n\to+\infty} A_{m,n} = 0$. Summing (3.98), we obtain

$$\sum_{i=m}^{m+k_0} \sum_{j=n}^{n+l_0} (A_{i+1,j} + A_{i,j+1} - A_{i,j}) + \sum_{i=m}^{m+k_0} \sum_{j=n}^{n+l_0} \sum_{t=1}^{u} P_t(i, j) f_t(A_{i-k_t,j-l_t}) = 0. \tag{3.112}$$

By Lemma 2.107,

$$A_{m,n} \geq \sum_{i=m}^{m+k_0} \sum_{j=n}^{n+l_0} \sum_{t=1}^{u} P_t(i, j) f_t(A_{i-k_t,j-l_t}). \tag{3.113}$$

Since f_i is monotone, we see that

$$A_{m,n} \geq \sum_{i=m}^{m+k_0} \sum_{j=n}^{n+l_0} \sum_{t=1}^{u} P_t(i, j) f_t(A_{m,n}) \tag{3.114}$$

or

$$\sum_{i=m}^{m+k_0}\sum_{j=n}^{n+l_0}\sum_{t=1}^{u} P_t(i,j)\frac{f_{t(A_{m,n})}}{A_{m,n}} \leq 1. \tag{3.115}$$

But this is contrary to (3.111). The proof is complete. □

Now we consider nonlinear partial difference equations of the form

$$A_{m-1,n} + A_{m,n-1} - A_{m,n} + \sum_{i=1}^{u} P_i(m,n) f_i(A_{m+k_i,n+l_i}) = 0, \tag{3.116}$$

where $P_i(m,n) \geq 0$, $k_i, l_i \in N_0$, $f_i \in C(R,R)$ and $xf_i(x) > 0$ for $x \neq 0$, $i = 1,2,\ldots,u$.

Theorem 3.32. *Every solution of* (3.116) *is oscillatory provided that the following conditions hold:*

(i) *for* $1 \leq i \leq u$,

$$\liminf_{x\to\infty} \frac{f_i(x)}{x} = H_i \in (0,\infty); \tag{3.117}$$

(ii) *for* $1 \leq i \leq u$,

$$\liminf_{m,n\to+\infty} P_i(m,n) = p_i > 0; \tag{3.118}$$

(iii) *for* $1 \leq i \leq u$, f_i *is nondecreasing;*

(iv)

$$\sum_{i=1}^{u} 2^{r_i} H_i p_i \frac{(r_i+1)^{r_i+1}}{r_i^{r_i}} > 1, \quad \textit{where } r_i = \min\{k_i, l_i\},\ i = 1,\ldots,u. \tag{3.119}$$

Proof. Suppose to the contrary, let $\{A_{m,n}\}$ be an eventually positive solution of (3.116). $A_{m,n}$ is increasing in m and n. Then we have $\lim_{m,n\to+\infty} A_{m,n} = k$.

We assert that $k = +\infty$. Otherwise, if k is finite, then by taking limits on both sides of (3.116) we have

$$k + \sum_{i=1}^{u} p_i f_i(k) \leq 0, \tag{3.120}$$

which is a contradiction.

In view of (3.116), we have

$$\frac{2A_{m-1,n-1}}{A_{m,n}} - 1 < \frac{A_{m,n-1} + A_{m-1,n}}{A_{m,n}} - 1 = -\sum_{i=1}^{u} P_i(m,n)\frac{f_i(A_{m+k_i,n+l_i})}{A_{m,n}} < -\sum_{i=1}^{u} P_i(m,n)\frac{f_i(A_{m+r_i,n+r_i})}{A_{m,n}}, \tag{3.121}$$

for all large m and n. Letting

$$\alpha_{m,n} = \frac{A_{m,n}}{A_{m-1,n-1}}, \tag{3.122}$$

we see that $\alpha_{m,n} > 1$ for all large m and n. The above inequality leads to

$$\frac{2}{\alpha_{m,n}} < 1 - \sum_{i=1}^{u} P_i(m,n)\frac{f_i(A_{m+r_i,n+r_i})}{A_{m+r_i,n+r_i}}\prod_{j=1}^{r_i}\alpha_{m+j,n+j}, \tag{3.123}$$

which implies that $\{\alpha_{m,n}\}$ is bounded. We now let $\xi = \liminf_{m,n\to+\infty}\alpha_{m,n}$. Then $\xi \in [1,+\infty)$. Furthermore, from the last inequality, we see that

$$\frac{2}{\xi} \le 1 - \sum_{i=1}^{u} p_i\xi^{r_i}H_i, \tag{3.124}$$

which implies $\xi > 2$ and

$$\sum_{i=1}^{u} p_iH_i\frac{\xi^{r_i+1}}{\xi-2} \le 1. \tag{3.125}$$

Note that

$$\min_{\xi>2}\frac{\xi^{r_i+1}}{\xi-2} = 2^{r_i}\frac{(r_i+1)^{r_i+1}}{r_i^{r_i}}, \tag{3.126}$$

thus we have

$$\sum_{i=1}^{u} p_iH_i2^{r_i}\frac{(r_i+1)^{r_i+1}}{r_i^{r_i}} \le 1, \tag{3.127}$$

which is contrary to assumption (iv). The proof is complete. □

Theorem 3.33. *Every solution of* (3.116) *is oscillatory if conditions (i) and (iii) of Theorem 3.32 hold and*

$$\limsup_{m,n\to+\infty}\sum_{t=1}^{u} H_t \sum_{i=m-k_0}^{m}\sum_{j=n-l_0}^{n} P_t(i,j) > 1, \tag{3.128}$$

where $k_0 = \min\{k_1, k_2, \dots, k_u\}$ *and* $l_0 = \min\{l_1, l_2, \dots, l_u\}$.

The proof is similar to the proof of Theorem 3.31.

3.3.2. Oscillation for the equation with continuous arguments

We consider nonlinear partial difference equations with continuous arguments of the form

$$A(x+a,y)+A(x,y+a)-A(x,y)+\sum_{i=1}^{m} h_i\big(x,y,A(x-\sigma_i,y-\tau_i)\big)=0, \tag{3.129}$$

where $h_i \in C(R^+ \times R^+ \times R, R)$, $uh_i(x,y,u) > 0$ for $u \neq 0$, h_i is nondecreasing in $u, a, \sigma_i, \tau_i > 0$, $i = 1,2,\dots,m$. Let $\sigma = \max_{1\le i\le m}\{\sigma_i\}$, $\tau = \max_{1\le i\le m}\{\tau_i\}$, $\sigma_i = k_i a + \zeta_i$, $\tau_i = l_i a + \xi_i$, where k_i, l_i are nonnegative integers, $\zeta_i, \xi_i \in [0,a)$.

Lemma 3.34. *Assume that* $A(x,y)$ *is an eventually positive solution of* (3.129). *Define*

$$Z(x,y) = \frac{1}{a^2}\int_x^{x+a}\int_y^{y+a} A(u,v)\,du\,dv, \tag{3.130}$$

then $Z(x,y) > 0$, $\partial Z/\partial x < 0$, $\partial Z/\partial y < 0$ *for all large* x *and* y.

Proof. Because $A(x,y)$ is an eventually positive solution of (3.129), $Z(x,y)>0$ eventually. From (3.129), we have $A(x+a,y)+A(x,y+a)-A(x,y) < 0$. Therefore

$$\frac{\partial Z}{\partial x} = \frac{1}{a^2}\int_y^{y+a}\big(A(x+a,v)-A(x,v)\big)dv < 0. \tag{3.131}$$

Similarly, $\partial Z/\partial y < 0$ eventually. □

Remark 3.35. Similar to Lemma 3.34, if $A(x,y)$ is an eventually negative solution of (3.129), then $Z(x,y) < 0$, $\partial Z/\partial x > 0$, $\partial Z/\partial y > 0$ eventually.

Theorem 3.36. *Every solution of* (3.129) *is oscillatory provided that the following conditions hold:*

(i) *for* $1 \le i \le m$,

$$\liminf_{x,y\to\infty,\,u\to 0} \frac{h_i(x,y,u)}{u} = S_i \ge 0, \qquad \sum_{i=1}^{m} S_i > 0; \tag{3.132}$$

(ii) *for* $1 \le i \le m$, $h_i(x,y,u)$ *is convex in* u *for* $u \ge 0$ *and concave in* u *for* $u < 0$;

(iii) *one of the following conditions holds:*

$$\sum_{i=1}^{m} 2^{\eta_i} S_i \frac{(\eta_i+1)^{\eta_i+1}}{\eta_i^{\eta_i}} > 1, \quad \eta_i = \min\{k_i, l_i\} > 0,\ i = 1,\dots,m; \tag{3.133}$$

$$\sum_{i=1}^{m} S_i \frac{k_i^{k_i}}{(k_i-1)^{k_i-1}} > 1 \quad \textit{if} \min_{1\le i\le m}\{k_i\} > 0, \quad \min_{1\le i\le m}\{l_i\} = 0; \tag{3.134}$$

$$\sum_{i=1}^{m} S_i \frac{l_i^{l_i}}{(l_i-1)^{l_i-1}} > 1 \quad \textit{if} \min_{1\le i\le m}\{k_i\} = 0, \quad \min_{1\le i\le m}\{l_i\} > 0; \tag{3.135}$$

$$\sum_{i=1}^{m} S_i > 1 \quad \textit{if} \min_{1\le i\le m}\{k_i\} = \min_{1\le i\le m}\{l_i\} = 0. \tag{3.136}$$

Proof. Suppose to the contrary that $A(x,y)$ is an eventually positive solution of (3.129). By Lemma 3.34, $\lim_{x,y\to+\infty} Z(x,y) = \zeta \ge 0$.

We claim that $\zeta = 0$. It is easy to see that

$$Z(x+a,y) + Z(x,y+a) - Z(x,y) + \sum_{i=1}^{m} h_i\big(x,y,Z(x-\sigma_i, y-\tau_i)\big) \le 0. \tag{3.137}$$

Hence

$$Z(x+a,y) + Z(x,y+a) - Z(x,y) \le 0. \tag{3.138}$$

If $\zeta > 0$, then by taking limits on both sides of the above inequality, we have $\zeta \le 0$, which is a contradiction. So $\zeta = 0$.

In view of (3.137), we have

$$\begin{aligned}
\frac{2Z(x+a,y+a)}{Z(x,y)}-1 &< \frac{Z(x+a,y)+Z(x,y+a)}{Z(x,y)}-1 \\
&\leq -\sum_{i=1}^{m}\frac{h_i(x,y,Z(x-\sigma_i,y-\tau_i))}{Z(x,y)} \\
&\leq -\sum_{i=1}^{m}\frac{h_i(x,y,Z(x-\eta_i a,y-\eta_i a))}{Z(x,y)} \\
&= -\sum_{i=1}^{m}\frac{h_i(x,y,Z(x-\eta_i a,y-\eta_i a))}{Z(x-\eta_i a,y-\eta_i a)} \\
&\quad \times\prod_{j=1}^{\eta_i}\frac{Z(x-ja,y-ja)}{Z(x-(j-1)a,y-(j-1)a)},
\end{aligned} \tag{3.139}$$

for all large x and y. Let

$$\alpha(x,y)=\frac{Z(x,y)}{Z(x+a,y+a)}. \tag{3.140}$$

Then $\alpha(x,y)>1$ for all large x and y. From the above inequality, we have

$$\frac{2}{\alpha(x,y)}+\sum_{i=1}^{m}\frac{h_i(x,y,Z(x-\eta_i a,y-\eta_i a))}{Z(x-\eta_i a,y-\eta_i a)}\prod_{j=1}^{\eta_i}\alpha(x-ja,y-ja)<1. \tag{3.141}$$

Condition (i) implies that $\alpha(x,y)$ is bounded. We rewrite (3.141) in the form

$$2+\sum_{i=1}^{m}\frac{h_i(x,y,Z(x-\eta_i a,y-\eta_i a))}{Z(x-\eta_i a,y-\eta_i a)}\prod_{j=1}^{\eta_i}\alpha(x-ja,y-ja)\alpha(x,y)<\alpha(x,y). \tag{3.142}$$

We now let $\beta=\liminf_{x,y\to+\infty}\alpha(x,y)$. Then $\beta\in[1,+\infty)$. Furthermore, from (3.142), taking the inferior limit on both sides, we obtain

$$2+\sum_{i=1}^{m}S_i\beta^{\eta_i}\beta\leq\beta. \tag{3.143}$$

Hence

$$\frac{2}{\beta}\leq 1-\sum_{i=1}^{m}S_i\beta^{\eta_i}, \tag{3.144}$$

which implies $\beta > 2$ and

$$\sum_{i=1}^{m} S_i \frac{\beta^{\eta_i+1}}{\beta - 2} \leq 1. \tag{3.145}$$

Note that

$$\min_{\beta>2} \frac{\beta^{\eta_i+1}}{\beta - 2} = 2^{\eta_i} \frac{(\eta_i + 1)^{\eta_i+1}}{\eta_i^{\eta_i}}, \tag{3.146}$$

thus we have

$$\sum_{i=1}^{m} S_i 2^{\eta_i} \frac{(\eta_i + 1)^{\eta_i+1}}{\eta_i^{\eta_i}} \leq 1, \tag{3.147}$$

which contradicts assumption (3.133).

For $\min_{1\leq i\leq m}\{k_i\} > 0$ and $\min_{1\leq i\leq m}\{l_i\} = 0$, from (3.137), we have

$$Z(x+a,y) + Z(x,y+a) - Z(x,y) + \sum_{i=1}^{m} h_i(x,y,Z(x-k_i a, y-\tau_i)) \leq 0. \tag{3.148}$$

Hence

$$\begin{aligned}
&\frac{Z(x+a,y) + Z(x,y+a)}{Z(x,y)} - 1 \\
&\quad \leq -\sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_i a, y-\tau_i))}{Z(x,y)} \\
&\quad = -\sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_i a, y-\tau_i))}{Z(x-k_i a, y-\tau_i)} \frac{Z(x-a,y-\tau_i)}{Z(x,y)} \prod_{j=2}^{k_i} \frac{Z(x-ja,y-\tau_i)}{Z(x-(j-1)a,y-\tau_i)}.
\end{aligned} \tag{3.149}$$

Since $Z(x,y)$ is decreasing in x and y, so $Z(x-a,y-\tau_i)/Z(x,y) > 1$, for all large x and y. The above inequality leads to

$$\frac{Z(x+a,y)}{Z(x,y)} + \sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_i a, y-\tau_i))}{Z(x-k_i a, y-\tau_i)} \prod_{j=2}^{k_i} \frac{Z(x-ja,y-\tau_i)}{Z(x-(j-1)a,y-\tau_i)} < 1. \tag{3.150}$$

Let $\alpha(x,y) = Z(x,y)/Z(x+a,y) > 1$. From (3.150), we have

$$\frac{1}{\alpha(x,y)} + \sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_i a, y-\tau_i))}{Z(x-k_i a, y-\tau_i)} \prod_{j=2}^{k_i} \alpha(x-ja,y-\tau_i) < 1. \tag{3.151}$$

(i) implies that $\alpha(x, y)$ is bounded. Let $\beta = \liminf_{x,y\to+\infty} \alpha(x, y)$. The above inequality leads to

$$\frac{1}{\beta} \leq 1 - \sum_{i=1}^{m} S_i \beta^{k_i - 1}, \tag{3.152}$$

which implies that $\beta > 1$ and

$$\sum_{i=1}^{m} S_i \frac{\beta^{k_i}}{\beta - 1} \leq 1. \tag{3.153}$$

Note that

$$\min_{\beta>1} \frac{\beta^{k_i}}{\beta - 1} = \frac{k_i^{k_i}}{(k_i - 1)^{k_i - 1}}, \tag{3.154}$$

we obtain

$$\sum_{i=1}^{m} S_i \frac{k_i^{k_i}}{(k_i - 1)^{k_i - 1}} \leq 1, \tag{3.155}$$

which contradicts (3.134).

The proof of (3.135) is similar to the proof of (3.134). Now we consider the last case, $\min_{1\leq i\leq m}\{k_i\} = \min_{1\leq i\leq m}\{l_i\} = 0$.

From (3.137),

$$\begin{aligned} &Z(x+a, y) + Z(x, y+a) - Z(x, y) + \sum_{i=1}^{m} h_i(x, y, Z(x, y)) \\ &\quad \leq Z(x+a, y) + Z(x, y+a) - Z(x, y) + \sum_{i=1}^{m} h_i\big(x, y, Z(x - \sigma_i, y - \tau_i)\big) \leq 0. \end{aligned} \tag{3.156}$$

Hence

$$\sum_{i=1}^{m} \frac{h_i(x, y, Z(x, y))}{Z(x, y)} - 1 \leq 0. \tag{3.157}$$

Taking the inferior limits on the above inequality, we have

$$\sum_{i=1}^{m} S_i - 1 \leq 0, \tag{3.158}$$

which contradicts (3.136). If $A(x, y)$ is an eventually negative solution of (3.129), we can lead to a contradiction by the similar method as the above. The proof is complete. □

Theorem 3.37. *Every solution of* (3.129) *is oscillatory if conditions (i) and (iii) of Theorem 3.36 hold and*

$$\limsup_{x,y\to+\infty,\,u\to 0} \sum_{n=1}^{m}\sum_{i=0}^{k_0}\sum_{j=0}^{l_0} \frac{h_n(x+ia, y+ja, u)}{u} > 1, \tag{3.159}$$

where $k_0 = \min\{k_1, k_2, \dots, k_m\}$ *and* $l_0 = \min\{l_1, l_2, \dots, l_m\}$.

Proof. Suppose to the contrary, let $A(x, y)$ be an eventually positive solution of (3.129). Then, as in the proof of Theorem 3.36, we have $\lim_{x,y\to+\infty} Z(x, y) = 0$ and

$$Z(x+a, y) + Z(x, y+a) - Z(x, y) + \sum_{i=1}^{m} h_i\big(x, y, Z(x-k_ia, y-l_ia)\big) \le 0. \tag{3.160}$$

Summing the above inequality, we obtain

$$\begin{aligned}\sum_{i=0}^{k_0}\sum_{j=0}^{l_0}&\big[Z(x+(i+1)a, y+ja) + Z(x+ia, y+(j+1)a) - Z(x+ia, y+ja)\big]\\ &+\sum_{i=0}^{k_0}\sum_{j=0}^{l_0}\left(\sum_{n=1}^{m} h_n\big(x+ia, y+ja, Z(x+(i-k_n)a, y+(j-l_n)a)\big)\right) \le 0.\end{aligned} \tag{3.161}$$

Similar to Lemma 2.107, the above inequality leads to

$$Z(x, y) \ge \sum_{i=0}^{k_0}\sum_{j=0}^{l_0}\sum_{n=1}^{m} h_n\big(x+ia, y+ja, Z(x+(i-k_n)a, y+(j-l_n)a)\big). \tag{3.162}$$

When $k_0 = 0$, we use σ_n to substitute k_na, when $l_0 = 0$, we use τ_n to substitute l_na. Since h_i is nondecreasing in u, we see that

$$Z(x, y) \ge \sum_{i=0}^{k_0}\sum_{j=0}^{l_0}\sum_{n=1}^{m} h_n\big(x+ia, y+ja, Z(x, y)\big), \tag{3.163}$$

which contradicts (3.159). In the case where $A(x, y)$ is eventually negative, the proof is similar to the above. The proof is complete. □

Example 3.38. Consider the partial difference equation

$$A(x+1, y)+A(x, y+1)-A(x, y)+p(x, y)\big(1+A^2(x-2, y-1)\big)A(x-2, y-1)=0, \tag{3.164}$$

where

$$p(x,y) = \frac{(y+2)(y-1)^3}{y(y+1)(y^2-2y+1+\sin^2(\pi x))},$$
$$h(x,y,u) = p(x,y)(1+u^2)u. \tag{3.165}$$

We see that $\liminf_{x,y\to\infty,\,u\to 0}(h(x,y,u)/u) = 1 > 0$ and

$$\limsup_{x,y\to\infty,\,|u|\to|b|} h(x,y,u\operatorname{sgn} u) = (1+b^2)|b| > 0 \quad \text{for } b \neq 0; \tag{3.166}$$

$h(x,y,u)$ is convex as $u \geq 0$ and concave as $u < 0$ and

$$\sum_{i=1}^{m} 2^{\eta_i} S_i \frac{(\eta_i+1)^{\eta_i+1}}{\eta_i^{\eta_i}} = 2(1+1)^{1+1} = 8 > 1, \tag{3.167}$$

so by Theorem 3.36, every solution of (3.164) is oscillatory. In fact, $A(x,y) = \sin(\pi x)/y$ is an oscillatory solution of (3.164).

3.3.3. Oscillation for the equation with mix nonlinear type

Consider nonlinear partial difference equations of the form

$$\begin{aligned} A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}\left|A_{m-k_1,n-l_1}\right|^{\alpha} \operatorname{sgn} A_{m-k_1,n-l_1} \\ + q_{m,n}\left|A_{m-k_2,n-l_2}\right|^{\beta} \operatorname{sgn} A_{m-k_2,n-l_2} = 0, \end{aligned} \tag{3.168}$$

where $p_{m,n} \geq 0$ and $q_{m,n} \geq 0$ on N_0^2, $k_1 \geq k_2 \geq 0$, $l_1 \geq l_2 \geq 0$, $l_i, k_i \in N_0$ for $i = 1, 2$, $\alpha \in [0,1)$, $\beta > 1$.

The following inequality will be used to prove the main result of this section.

Lemma 3.39. *Let* $x, y \geq 0$, $m, n > 1$ *and* $1/m + 1/n = 1$. *Then*

$$\frac{x}{m} + \frac{y}{n} \geq x^{1/m} y^{1/n}. \tag{3.169}$$

Define the subset of the positive reals as follows:

$$E = \left\{\lambda > 0 \mid 1 - \lambda p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} > 0 \text{ eventually}\right\}. \tag{3.170}$$

Given an eventually positive solution $\{A_{m,n}\}$ of (3.168), define the subset $S(A)$ of the positive reals as follows:

$$S(A) = \left\{\lambda > 0 \mid A_{m+1,n} + A_{m,n+1} - A_{m,n}\left(1 - \lambda p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)}\right) \leq 0 \text{ eventually}\right\}. \tag{3.171}$$

If $\lambda \in S(A)$, then $1 - \lambda p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} > 0$ eventually. Therefore $S(A) \subset E$.

It is easy to see that condition

$$\limsup_{m,n\to\infty} p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} > 0 \tag{3.172}$$

implies that the set E is bounded.

Theorem 3.40. *Assume that*
(i) (3.172) *holds;*
(ii)

$$\sup_{\lambda\in E,\, m\ge M,\, n\ge N} \lambda\theta\left\{\prod_{i=m-k_2}^{m-1}\prod_{j=n-l_2}^{n-1}\left(1-\lambda p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right)\right\}^{1/\eta} < 1, \tag{$*$}$$

where $\eta = \min\{k_2, l_2\} \ge 1$, $\theta = \min\{(\beta-\alpha)/(\beta-1), (\beta-\alpha)/(1-\alpha)\} > 1$, M, N *are large integers. Then every solution of* (3.168) *oscillates.*

Proof. Suppose to the contrary, let $\{A_{i,j}\}$ be an eventually positive solution of (3.168). Then $A_{m,n}$ is decreasing in m, n. Hence we have

$$\begin{gathered} A_{m-k_1,n-l_1} \ge A_{m-k_2,n-l_2}, \\ A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n}A_{m-k_2,n-l_2}^{\alpha} + q_{m,n}A_{m-k_2,n-l_2}^{\beta} \le 0. \end{gathered} \tag{3.173}$$

By Lemma 3.39, we have

$$p_{m,n}A_{m-k_2,n-l_2}^{\alpha} + q_{m,n}A_{m-k_2,n-l_2}^{\beta} \ge \theta p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} A_{m-k_2,n-l_2}. \tag{3.174}$$

From (3.173) and (3.174), we obtain

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \theta p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} A_{m-k_2,n-l_2} \le 0, \tag{3.175}$$

thus we have

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \theta p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} A_{m,n} \le 0, \tag{3.176}$$

so

$$0 < A_{m+1,n} + A_{m,n+1} \le \left(1-\theta p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)}\right) A_{m,n}, \tag{3.177}$$

which implies that $S(A)$ is nonempty. Let $\mu \in S(A)$, then

$$A_{m+1,n} \le \left(1-\mu p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)}\right) A_{m,n} \tag{3.178}$$

and so

$$A_{m,n} \leq \prod_{i=m-k_2}^{m-1} \left(1 - \mu p_{i,n}^{(\beta-1)/(\beta-\alpha)} q_{i,n}^{(1-\alpha)/(\beta-\alpha)}\right) A_{m-k_2,n}. \tag{3.179}$$

Similarly, we have

$$A_{m,n+1} \leq \left(1 - \mu p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)}\right) A_{m,n} \tag{3.180}$$

and so

$$A_{m,n} \leq \prod_{j=n-l_2}^{n-1} \left(1 - \mu p_{m,j}^{(\beta-1)/(\beta-\alpha)} q_{m,j}^{(1-\alpha)/(\beta-\alpha)}\right) A_{m,n-l_2}. \tag{3.181}$$

Hence

$$A_{m,n}^{l_2} \leq A_{m,n-1} \cdots A_{m,n-l_2} \leq \prod_{j=n-l_2}^{n-1} \prod_{i=m-k_2}^{m-1} \left(1 - \mu p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right) A_{m-k_2,n-l_2}^{l_2}. \tag{3.182}$$

Similarly, we have

$$A_{m,n}^{k_2} \leq \prod_{i=m-k_2}^{m-1} \prod_{j=n-l_2}^{n-1} \left(1 - \mu p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right) A_{m-k_2,n-l_2}^{k_2}. \tag{3.183}$$

Combining (3.182) and (3.183), we obtain

$$A_{m,n} \leq \left\{ \prod_{i=m-k_2}^{m-1} \prod_{j=n-l_2}^{n-1} \left(1 - \mu p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right) \right\}^{1/\eta} A_{m-k_2,n-l_2}. \tag{3.184}$$

Substituting (3.184) into (3.175), we obtain

$$\begin{aligned} A_{m+1,n} + A_{m,n+1} - A_{m,n} \Bigg\{ 1 - \theta p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} \\ \times \left[\prod_{i=m-k_2}^{m-1} \prod_{j=n-l_2}^{n-1} \left(1 - \mu p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right) \right]^{-1/\eta} \Bigg\} \leq 0, \end{aligned} \tag{3.185}$$

which implies that

$$\theta \left\{ \sup_{m \geq M, n \geq N} \left[\prod_{i=m-k_2}^{m-1} \prod_{j=n-l_2}^{n-1} \left(1 - \mu p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right) \right]^{1/\eta} \right\}^{-1} \in S(A). \tag{3.186}$$

From condition (ii), there exists $\gamma \in (0,1)$ such that

$$\sup_{\lambda\in E,\, m\geq M,\, n\geq N} \lambda\theta\left\{\prod_{i=m-k_2}^{m-1}\prod_{j=n-l_2}^{n-1}\left(1-\lambda p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right)\right\}^{1/\eta} \leq \gamma < 1. \tag{3.187}$$

Hence

$$\left\{\sup_{m\geq M,\, n\geq N}\left[\prod_{i=m-k_2}^{m-1}\prod_{j=n-l_2}^{n-1}\left(1-\mu p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right)\right]^{1/\eta}\right\}^{-1} \geq \frac{\mu}{\gamma}, \tag{3.188}$$

so that $\mu/\gamma \in S(A)$. By induction, $\mu/\gamma^r \in S(A)$, $r = 1,2,\ldots$. This contradicts the boundedness of $S(A)$. The proof is complete. □

From Theorem 3.40, we can derive an explicit oscillation criterion.

Corollary 3.41. *Assume that*

$$\liminf_{m,n\to\infty} \frac{1}{k_2 l_2}\sum_{i=m-k_2}^{m-1}\sum_{j=n-l_2}^{n-1} p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)} > \frac{\theta a^a}{(1+a)^{1+a}}, \tag{3.189}$$

where $a = \max\{k_2, l_2\}$. *Then every solution of* (3.168) *is oscillatory.*

Proof. Let $g(\lambda) = \lambda(1-c\lambda)^a$ for $\lambda > 0$, $c > 0$. Then

$$\max_{\lambda>0} g(\lambda) = \frac{a^a}{c(1+a)^{1+a}}. \tag{3.190}$$

Set

$$c = \frac{1}{k_2\, l_2}\sum_{i=m-k_2}^{m-1}\sum_{j=n-l_2}^{n-1} p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}. \tag{3.191}$$

Since

$$\begin{aligned}\left\{1-\frac{\lambda}{k_2 l_2}\sum_{i=m-k_2}^{m-1}\sum_{j=n-l_2}^{n-1} p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right\}^{a}\\ \geq \left\{\prod_{i=m-k_2}^{m-1}\prod_{j=n-l_2}^{n-1}\left(1-\lambda p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right)\right\}^{1/\eta},\end{aligned} \tag{3.192}$$

we obtain

$$\begin{aligned}1 &> \theta\frac{a^a}{(1+a)^{1+a}}\left\{\frac{1}{k_2l_2}\sum_{i=m-k_2}^{m-1}\sum_{j=n-l_2}^{n-1}p_{i,j}^{(\beta-1)/(\beta-\alpha)}q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right\}^{-1}\\ &\geq \lambda\theta\left(1-\frac{\lambda}{k_2l_2}\sum_{i=m-k_2}^{m-1}\sum_{j=n-l_2}^{n-1}p_{i,j}^{(\beta-1)/(\beta-\alpha)}q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right)^a\\ &\geq \lambda\theta\left\{\prod_{i=m-k_2}^{m-1}\prod_{j=n-l_2}^{n-1}\left(1-\lambda p_{i,j}^{(\beta-1)/(\beta-\alpha)}q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right)\right\}^{1/\eta}.\end{aligned} \tag{3.193}$$

Then the conclusion follows from Theorem 3.40. □

Example 3.42. Consider the equation

$$\begin{aligned}A_{m+1,n}+A_{m,n+1}-A_{m,n}+p_{m,n}\left|A_{m-4,n-2}\right|^{\alpha}\operatorname{sgn}A_{m-4,n-2}&\\ +q_{m,n}\left|A_{m-1,n-1}\right|^{\beta}\operatorname{sgn}A_{m-1,n-1}&=0,\end{aligned} \tag{3.194}$$

where $\alpha = 1/8$, $\beta = 9/8$, $p_{m,n} = e^{-(7/8)n-5/4}(e-1) \geq 0$, $q_{m,n} = e^{(1/8)n-9/8} \geq 0$, $\theta = \min\{(\beta-\alpha)/(\beta-1), (\beta-\alpha)/(1-\alpha)\} = 8/7$, and $a = \max\{k_2, l_2\} = 1$, so

$$\liminf_{m,n\to\infty}\frac{1}{k_2l_2}\sum_{i=m-k_2}^{m-1}\sum_{j=n-l_2}^{n-1}p_{i,j}^{(\beta-1)/(\beta-\alpha)}q_{i,j}^{(1-\alpha)/(\beta-\alpha)} = (e-1)^{1/8}e^{-73/64} > \frac{\theta a^a}{(1+a)^{1+a}} = \frac{2}{7}. \tag{3.195}$$

Hence every solution of (3.194) oscillates. In fact, $e^{-n}\sin(\pi/2)m$ is an oscillatory solution.

If $q_{m,n} \equiv 0$, (3.168) becomes the sublinear equation

$$A_{m+1,n}+A_{m,n+1}-A_{m,n}+p_{m,n}\left|A_{m-k,n-l}\right|^{\alpha}\operatorname{sgn}A_{m-k,n-l}=0, \tag{3.196}$$

where $0 < \alpha < 1$.

Theorem 3.43. *Assume that* $p_{m,n} \geq 0$ *and*

$$\sum_{(i,j)=(m,n)}^{\infty}p_{i,j}=\infty. \tag{3.197}$$

Then every solution of (3.196) *oscillates.*

Proof. Suppose $\{A_{m,n}\}$ is an eventually positive solution of (3.196). Then $A_{m,n}$ is decreasing in m, n eventually. Hence $A_{m,n} \to L \geq 0$ as $m, n \to \infty$.

Summing (3.196) in n from $n(\geq N)$ to ∞, we have

$$\sum_{i=n}^{\infty} A_{m+1,i} - A_{m,n} + \sum_{i=n}^{\infty} p_{m,i} A^{\alpha}_{m-k,i-l} \leq 0, \tag{3.198}$$

we rewrite the above equation in the form

$$\sum_{i=n+1}^{\infty} A_{m+1,i} + A_{m+1,n} - A_{m,n} + \sum_{i=n}^{\infty} p_{m,i} A^{\alpha}_{m-k,i-l} \leq 0. \tag{3.199}$$

Summing (3.199) in m from $m(\geq M)$ to ∞, we obtain

$$-A_{m,n} + \sum_{j=m}^{\infty} \sum_{i=n+1}^{\infty} A_{j+1,i} + \sum_{j=m}^{\infty} \sum_{i=n+1}^{\infty} p_{j,i} A^{\alpha}_{j-k,i-l} \leq 0. \tag{3.200}$$

Thus

$$A_{m,n} \geq \sum_{j=m}^{\infty} \sum_{i=n}^{\infty} p_{j,i} A^{\alpha}_{j-k,i-l}, \tag{3.201}$$

which contradicts (3.197) if $L > 0$.

If $L = 0$, then from (3.197), we can see that

$$\infty = \sum_{(i,j)=(m,n)}^{\infty} p_{i,j} \leq \sum_{(i,j)=(m,n)}^{\infty} p_{i,j} \frac{A^{\alpha}_{i-k,j-l}}{A^{\alpha}_{i,j}} \leq \sum_{(i,j)=(m,n)}^{\infty} \frac{A_{i,j}}{A^{\alpha}_{i,j}} = \sum_{(i,j)=(m,n)}^{\infty} A^{1-\alpha}_{i,j}. \tag{3.202}$$

Note that $\gamma = 1 - \alpha$, then $0 < \gamma < 1$. Notice that

$$A_{i,j} > A_{i+1,j} + A_{i,j+1} > A_{i+1,j+1} + A_{i+1,j+1} = 2A_{i+1,j+1}, \tag{3.203}$$

we can get $A_{i+1,j+1} < (1/2)A_{i,j}$. Thus

$$\begin{aligned} \sum_{(i,j)=(m,n)}^{\infty} A^{1-\alpha}_{i,j} &= \sum_{(i,j)=(m,n)}^{\infty} A^{\gamma}_{i,j} = \sum_{i=m}^{\infty} \sum_{\bar{k}=0}^{\infty} A^{\gamma}_{i+\bar{k},n+\bar{k}} + \sum_{j=n+1}^{\infty} \sum_{\bar{k}=0}^{\infty} A^{\gamma}_{m+\bar{k},j+\bar{k}} \\ &< \sum_{i=m}^{\infty} \frac{2^{\gamma}}{2^{\gamma}-1} A^{\gamma}_{i,n} + \sum_{j=n+1}^{\infty} \frac{2^{\gamma}}{2^{\gamma}-1} A^{\gamma}_{m,j}. \end{aligned} \tag{3.204}$$

So, if we can show that $\sum_{i=m}^{\infty} A^{\gamma}_{i,n}$ and $\sum_{j=n+1}^{\infty} A^{\gamma}_{m,j}$ converge, the conclusion can be drew naturally.

We only discuss the series $\sum_{j=n+1}^{\infty} A^{\gamma}_{m,j}$ and the next case is similar.

Note that $J_j = \{\bar{j} \in Z \mid A_{m,j+1} \le A^{\gamma}_{m,\bar{j}} < A_{m,j}\}$ for $j = n, n+1, \ldots$. We can see for all $j = n, n+1, \ldots, J$ is finite. Let $J_1 = \max_{j \ge n} |J_j|$, where $|J_i|$ denotes the number of the elements in it, and $j_1 = \min\{\bar{j} \mid \bar{j} \in J_n\}$, so

$$\sum_{j=n}^{\infty} A^{\gamma}_{m,j} \le J_1 \sum_{j=j_1}^{\infty} A_{m,j} + \sum_{j=n}^{j_1-1} A^{\gamma}_{m,j}. \tag{3.205}$$

Then the series $\sum_{j=n}^{\infty} A^{\gamma}_{m,j}$ converges to a constant according to (3.200), which contradicts to (3.202).

If $p_{m,n} \equiv 0$, (3.168) becomes the superlinear equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + q_{m,n} \left| A_{m-k_2,n-l_2} \right|^{\beta} \operatorname{sgn} A_{m-k_2,n-l_2} = 0, \tag{3.206}$$

where $\beta > 1$. □

Theorem 3.44. *If*

$$0 < q_{m,n} \le \frac{k_2^{k_2} l_2^{l_2}}{(k_2 + l_2 + 1)^{k_2+l_2+1}}, \tag{3.207}$$

then (3.206) *has an eventually positive solution.*

In fact, Theorem 3.44 follows from Theorem 3.7.

In the following, the result of Theorem 3.40 will be improved.

Theorem 3.45. *Assume that (i) holds and*

$$\sup_{\lambda \in E,\, m \ge M,\, n \ge N} \lambda\theta \left\{ \prod_{i=m-k_2}^{m-1} \prod_{j=n-l_2}^{n-1} \left(1 - \theta p^{(\beta-1)/(\beta-\alpha)}_{i+\eta,j+\eta} q^{(1-\alpha)/(\beta-\alpha)}_{i+\eta,j+\eta} - \lambda p^{(\beta-1)/(\beta-\alpha)}_{i,j} q^{(1-\alpha)/(\beta-\alpha)}_{i,j}\right) \right\}^{1/\eta} < 1, \tag{3.208}$$

where $\eta = \min\{k_2, l_2\}$. *Then every solution of* (3.168) *is oscillatory.*

Proof. Suppose to the contrary, let $\{A_{i,j}\}$ be an eventually positive solution of (3.168). As in the proof of Theorem 3.40, set $\mu \in S(A)$. Then

$$A_{m+1,n} + A_{m,n+1} \le \left(1 - \mu p^{(\beta-1)/(\beta-\alpha)}_{m,n} q^{(1-\alpha)/(\beta-\alpha)}_{m,n}\right) A_{m,n}. \tag{3.209}$$

Hence

$$\begin{aligned} A_{m+1,n} &\le \left(1 - \mu p^{(\beta-1)/(\beta-\alpha)}_{m,n} q^{(1-\alpha)/(\beta-\alpha)}_{m,n}\right) A_{m,n} - A_{m,n+1} \\ &\le \left(1 - \mu p^{(\beta-1)/(\beta-\alpha)}_{m,n} q^{(1-\alpha)/(\beta-\alpha)}_{m,n}\right) A_{m,n} - A_{m+\eta,n+\eta}. \end{aligned} \tag{3.210}$$

From (3.174), we have

$$\theta p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} A_{m-\eta,n-\eta} \le \theta p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} A_{m-k_2,n-l_2} \le A_{m,n}. \tag{3.211}$$

Substituting (3.211) into (3.210), we obtain

$$A_{m+1,n} \le \left(1 - \mu p_{m,n}^{(\beta-1)/(\beta-\alpha)} q_{m,n}^{(1-\alpha)/(\beta-\alpha)} - \theta p_{m+\eta,n+\eta}^{(\beta-1)/(\beta-\alpha)} q_{m+\eta,n+\eta}^{(1-\alpha)/(\beta-\alpha)}\right) A_{m,n}. \tag{3.212}$$

By the similar argument of the proof of Theorem 3.40, we can derive

$$\begin{aligned} A_{m,n} \le \Bigg\{ \prod_{i=m-k_2}^{m-1} \prod_{j=n-l_2}^{n-1} \Big(1 - \mu p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)} \\ - \theta p_{i+\eta,j+\eta}^{(\beta-1)/(\beta-\alpha)} q_{i+\eta,j+\eta}^{(1-\alpha)/(\beta-\alpha)}\Big)\Bigg\}^{1/\eta} A_{m-k_2,n-l_2}. \end{aligned} \tag{3.213}$$

Replacing (3.184) by (3.213), the rest of the proof is exactly the same with the proof of Theorem 3.40. The proof is complete. □

Corollary 3.46. *Assume that (i) holds and*

$$\liminf_{m,n\to\infty} \frac{1}{k_2 l_2} \sum_{i=m-k_2}^{m-1} \sum_{j=n-l_2}^{n-1} p_{i,j}^{(\beta-1)/(\beta-\alpha)} q_{i,j}^{(1-\alpha)/(\beta-\alpha)} > \theta \frac{a^a}{(1+a)^{1+a}} (1 - b\theta)^{a+1}, \tag{3.214}$$

where

$$\liminf_{m,n\to\infty} \frac{1}{k_2 l_2} \sum_{i=m-k_2}^{m-1} \sum_{j=n-l_2}^{n-1} p_{i+\eta,j+\eta}^{(\beta-1)/(\beta-\alpha)} q_{i+\eta,j+\eta}^{(1-\alpha)/(\beta-\alpha)} = b. \tag{3.215}$$

Then every solution of (3.168) *oscillates.*

The main idea of Theorem 3.45 is to improve the estimation (3.184). Therefore this method is also available for the linear equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + p_{m,n} A_{m-k,n-l} = 0. \tag{3.216}$$

Theorem 3.47. *Assume that* $\eta = \min\{k, l\}$, $\limsup_{m,n\to\infty} p_{m,n} > 0$, *and*

$$\sup_{\lambda\in E,\, m\ge M,\, n\ge N} \lambda \Bigg\{ \prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} \left(1 - \lambda p_{i,j} - p_{i+\eta,j+\eta}\right) \Bigg\}^{1/\eta} < 1. \tag{3.217}$$

Then every solution of (3.216) *oscillates.*

Corollary 3.48. *If* (3.217) *is replaced by*

$$\liminf_{m,n\to\infty}\frac{1}{kl}\sum_{i=m-k}^{m-1}\sum_{j=n-l}^{n-1}p_{i,j}>(1-d)^{1+a}\frac{a^a}{(1+a)^{1+a}}, \tag{3.218}$$

where $a=\max\{k,l\}$ *and*

$$d=\liminf_{m,n\to\infty}\frac{1}{kl}\sum_{i=m-k}^{m-1}\sum_{j=n-l}^{n-1}p_{i+\eta,j+\eta}. \tag{3.219}$$

Then every solution of (3.216) *oscillates.*

Remark 3.49. Equations (3.217) and (3.218) improve the corresponding results in Chapter 2.

Remark 3.50. If $k_1 \ge k_2 \ge 0$ and $0 \le l_1 \le l_2$, then (3.174) becomes

$$A_{m+1,n}+A_{m,n+1}-A_{m,n}+\theta p_{m,n}^{(\beta-1)/(\beta-\alpha)}q_{m,n}^{(1-\alpha)/(\beta-\alpha)}A_{m-k_2,n-l_1}\le 0, \tag{3.220}$$

(ii) becomes

$$\sup_{\lambda\in E,\, m\ge M,\, n\ge N}\lambda\theta\left\{\prod_{i=m-k_2}^{m-1}\prod_{j=n-l_1}^{n-1}\left(1-\lambda p_{i,j}^{(\beta-1)/(\beta-\alpha)}q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right)\right\}^{1/\eta}<1, \tag{3.221}$$

where $\eta=\min\{k_2,l_1\}$. Then Theorem 3.40 is also true.

Similarly, we can easily derive the form of Theorem 3.40 for the case that $0\le k_1\le k_2$ and $l_1\ge l_2\ge 0$.

If $0\le k_1\le k_2$ and $0\le l_1\le l_2$, then (3.174) becomes

$$A_{m+1,n}+A_{m,n+1}-A_{m,n}+\theta p_{m,n}^{(\beta-1)/(\beta-\alpha)}q_{m,n}^{(1-\alpha)/(\beta-\alpha)}A_{m-k_1,n-l_1}\le 0, \tag{3.222}$$

and (ii) becomes

$$\sup_{\lambda\in E,\, m\ge M,\, n\ge N}\lambda\theta\left\{\prod_{i=m-k_1}^{m-1}\prod_{j=n-l_1}^{n-1}\left(1-\lambda p_{i,j}^{(\beta-1)/(\beta-\alpha)}q_{i,j}^{(1-\alpha)/(\beta-\alpha)}\right)\right\}^{1/\eta}<1, \tag{3.223}$$

where $\eta=\min\{k_1,l_1\}$. Then Theorem 3.40 is also true.

Example 3.51. Consider the equation

$$A_{m+1,n}+A_{m,n+1}-A_{m,n}+A_{m-1,n-2}=0, \tag{3.224}$$

we can see that (3.218) is satisfied, so every solution of this equation oscillates. In fact, $\sin(\pi/2)m$ is an oscillatory solution.

Next, we consider (3.168) when $k_2 = 0$, $l_2 > 0$ or $k_2 > 0$, $l_2 = 0$ and we can get the following conclusion.

Theorem 3.52. *If* $k_2 = 0$, $l_2 > 0$ *in* (3.168). *Assume that* (3.172) *holds and*

$$\sup_{\lambda \in E,\, m \geq M,\, n \geq N} \lambda\theta \prod_{j=n-l_2}^{n-1} \left(1 - \lambda p_{m,j}^{(\beta-1)/(\beta-\alpha)} q_{m,j}^{(1-\alpha)/(\beta-\alpha)}\right) < 1. \tag{3.225}$$

Then every solution of (3.168) *oscillates.*

From (3.225), we can drive an explicit oscillation condition.

Corollary 3.53. *Assume that* $k_2 = 0$, $l_2 > 0$ *in* (3.168). *Further assume that* (3.172) *holds and*

$$\liminf_{m,n \to \infty} \frac{1}{l_2} \sum_{j=n-l_2}^{n-1} p_{m,j}^{(\beta-1)/(\beta-\alpha)} q_{m,j}^{(1-\alpha)/(\beta-\alpha)} > \frac{\theta l_2^{l_2}}{(1+l_2)^{1+l_2}}. \tag{3.226}$$

Then every solution of (3.168) *oscillates.*

When $k_2 > 0$, $l_2 = 0$, the similar conclusion holds, we omit it in detail here.

By using the inequality

$$\sum_{i=1}^{u} \alpha_i x_i \geq \prod_{i=1}^{u} x_i^{\alpha_i}, \tag{3.227}$$

where $\alpha_i > 0$, $\sum_{i=1}^{u} \alpha_i = 1$, $x_i \geq 0$, $i = 1, 2, \ldots, u$, we can consider the partial difference equation with several nonlinear terms of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} P_i(m,n) \left| A_{m-k_i,n-l_i} \right|^{\alpha_i} \operatorname{sgn} A_{m-k_i,n-l_i} = 0, \tag{3.228}$$

where $\alpha_u > \alpha_{u-1} > \cdots > \alpha_k > 1 > \alpha_{k-1} > \cdots > \alpha_1 > 0$, $P_i(m,n) \geq 0$, $i = 1, 2, \ldots, u$ on N_0^2, $k_i, l_i \in N_0$, $i = 1, 2, \ldots, u$.

Theorem 3.54. *Assume that there exist* $a_1 > 0$, $a_2 > 0, \ldots, a_u > 0$ *such that* $\sum_{i=1}^{u} a_i = 1$, $\sum_{i=1}^{u} a_i \alpha_i = 1$,

$$\begin{gathered} \limsup_{m,n \to \infty} \prod_{i=1}^{u} P_i^{a_i}(m,n) > 0, \\ \sup_{\lambda \in E,\, m \geq M,\, n \geq N} \lambda\theta \left\{ \prod_{i=m-\bar{k}}^{m-1} \prod_{j=n-\bar{l}}^{n-1} \left(1 - \lambda \prod_{h=1}^{u} P_h^{a_h}(i,j)\right) \right\}^{1/\eta} < 1, \end{gathered} \tag{3.229}$$

where $\bar{k} = \min_{1\le i\le u}\{k_i\}$, $\bar{l} = \min_{1\le i\le u}\{l_i\}$, $\eta = \min\{\bar{k}, \bar{l}\} \ge 1$, $\theta = \min\{1/a_1, \dots, 1/a_u\}$,

$$E = \left\{\lambda > 0 \mid 1 - \lambda \prod_{h=1}^{u} P_h^{a_h}(m,n) > 0 \text{ eventually}\right\}, \tag{3.230}$$

M, N are large integers. Then every solution of (3.228) *oscillates.*

Proof. Suppose to the contrary, let $\{A_{i,j}\}$ be an eventually positive solution of (3.228). Then $A_{m,n}$ is decreasing in m, n. Hence

$$A_{m-k_i,n-l_i} > A_{m-\bar{k},n-\bar{l}}, \quad i = 1, 2, \dots, u. \tag{3.231}$$

From (3.227), we have

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} P_i(m,n) A_{m-\bar{k},n-\bar{l}}^{\alpha_i} \le 0. \tag{3.232}$$

Using (3.227) and (3.232), we have

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \theta \prod_{i=1}^{u} P_i^{a_i}(m,n) A_{m-\bar{k},n-\bar{l}} \le 0. \tag{3.233}$$

The rest of the proof is similar to those in Theorem 3.40. We omit it in detail. The proof is complete. □

For example, we consider the case $u = 3$, $\alpha_3 > 1 > \alpha_2 > \alpha_1 > 0$. Let

$$\begin{aligned} a_1 &= \frac{\alpha_3 - 1}{2(\alpha_3 - \alpha_1)}, \qquad a_2 = \frac{\alpha_3 - 1}{2(\alpha_3 - \alpha_2)}, \\ a_3 &= \frac{2(1-\alpha_2)(\alpha_3-\alpha_1) + (\alpha_3 - 1)(\alpha_2 - \alpha_1)}{2(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}. \end{aligned} \tag{3.234}$$

Then $a_i > 0$, $i = 1, 2, 3$, $\sum_{i=1}^{3} a_i = 1$, $\sum_{i=1}^{u} \alpha_i a_i = 1$.

Theorem 3.55. *Assume that*

$$\limsup_{m,n\to\infty} \prod_{i=1}^{3} P_i^{a_i}(m,n) > 0, \tag{3.235}$$

$$\sup_{\lambda\in E,\, m\ge M,\, n\ge N} \lambda\theta \left\{ \prod_{i=m-\bar{k}}^{m-1} \prod_{j=n-\bar{l}}^{n-1} \left(1 - \lambda \prod_{h=1}^{3} P_h^{a_h}(i,j)\right)\right\}^{1/\eta} < 1, \tag{3.236}$$

where $\bar{k} = \min\{k_1, k_2, k_3\}$, $\bar{l} = \min\{l_1, l_2, l_3\}$, $\eta = \min\{\bar{k}, \bar{l}\} \geq 1$, $\theta = \min\{1/a_1, 1/a_2, 1/a_3\}$, *M, N are large integers and*

$$E = \left\{\lambda > 0 \mid 1 - \lambda \prod_{i=1}^{3} P_h^{a_h}(i,j) > 0 \text{ eventually}\right\}. \tag{3.237}$$

Then every solution of the equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{3} P_i(m,n) \left| A_{m-k_i,n-l_i} \right|^{\alpha_i} \operatorname{sgn} A_{m-k_i,n-l_i} = 0 \tag{3.238}$$

is oscillatory.

From (3.236), we can obtain the following explicit oscillation criterion.

Corollary 3.56. *Assume that* (3.235) *holds. Further, assume that*

$$\liminf_{m,n\to\infty} \frac{1}{k_3 l_3} \sum_{i=m-\bar{k}}^{m-1} \sum_{n-\bar{l}}^{n-1} P_1^{a_1}(i,j) P_2^{a_2}(i,j) P_3^{a_3}(i,j) > \frac{\theta a^a}{(1+a)^{1+a}}, \tag{3.239}$$

where $a = \max\{\bar{k}, \bar{l}\} \geq 1$. *Then every solution of* (3.238) *oscillates.*

3.4. Existence of oscillatory solutions of certain nonlinear PDEs

Consider the partial difference equation

$$\Delta_m^h \Delta_n^r (x_{m,n} - c x_{m-k,n-l}) + f(m,n,x_{m-\tau,n-\sigma}) = 0, \quad m,n \in N_0, \tag{3.240}$$

where $c \neq 0$ is a real constant, $h, r, k, l \in N_1$, $\tau, \sigma \in N$, Δ is the forward difference operator defined by $\Delta_m x_{m,n} = x_{m+1,n} - x_{m,n}$, $\Delta_n x_{m,n} = x_{m,n+1} - x_{m,n}$ and $\Delta_m^h x_{m,n} = \Delta_m(\Delta_m^{h-1} x_{m,n})$, $\Delta_m^0 x_{m,n} = x_{m,n}$, $\Delta_n^r x_{m,n} = \Delta_n(\Delta_n^{r-1} x_{m,n})$, $\Delta_n^0 x_{m,n} = x_{m,n}$, $f \in C(N_0 \times N_0 \times R, R)$. Throughout the section, we assume that there exists a continuous function $F : N_0 \times N_0 \times [0,\infty) \to [0,\infty)$ such that $F(m,n,u)$ is nondecreasing in u and

$$\left| f(m,n,u) \right| \leq F(m,n,|u|), \quad (m,n,u) \in N_0 \times N_0 \times R. \tag{3.241}$$

A solution of (3.240) is a real double sequence defined for all $(m,n) \in \{(m,n) \mid m \geq \min\{M-k, M-\tau\},\ n \geq \min\{N-l, N-\sigma\}\}$ and satisfying (3.240) for all $(m,n) \in \{(m,n) \mid m \geq M,\ n \geq N\}$, where $M, N \in N_0$.

The definition of oscillatory solutions of (3.240) is same with Chapter 2.

Let X be the linear space of all bounded real sequences $x = \{x_{m,n}\}$, $m \geq M$, $n \geq N$ endowed with the usual norm

$$\|x\| = \sup_{m\geq M,\, n\geq N} |x_{m,n}|, \tag{3.242}$$

then X is a Banach space.

Let Ω be a subset of Banach space X. Ω is relatively compact if every sequence in Ω has a subsequence converging to an element of X. An ϵ-net for Ω is a set of elements of X such that each x in Ω is within a distance ϵ of some member of the net. A finite ϵ-net is an ϵ-net consisting of a finite number of the elements.

Lemma 3.57. *A subset Ω of a Banach space X is relatively compact if and only if for each $\epsilon > 0$, it has a finite ϵ-net.*

Definition 3.58. A set Ω of Banach space X is uniformly Cauchy if for every $\epsilon > 0$ there exist positive integers M_1 and N_1 such that for any $x = \{x_{m,n}\}$ in Ω

$$|x_{m,n} - x_{m',n'}| < \epsilon, \tag{3.243}$$

whenever $(m,n) \in D'$, $(m',n') \in D'$, where $D' = D'_1 \cup D'_2 \cup D'_3$,

$$D'_1 = \{(m,n) \mid m > M_1,\ n > N_1\}, \qquad D'_2 = \{(m,n) \mid M \leq m \leq M_1,\ n > N_1\},$$
$$D'_3 = \{(m,n) \mid m > M_1,\ N \leq n \leq N_1\}. \tag{3.244}$$

Lemma 3.59 (Discrete Arzela-Ascoli's theorem). *A bounded, uniformly Cauchy subset Ω of X is relatively compact.*

Proof. By Lemma 3.57, it suffices to construct a finite ϵ-net for any $\epsilon > 0$. We know that for any $\epsilon > 0$, there are integers M_1 and N_1 such that for any $x \in \Omega$

$$|x_{m,n} - x_{m',n'}| < \frac{\epsilon}{2} \quad \text{for } (m,n) \in D',\ (m',n') \in D'. \tag{3.245}$$

Let K be a bound of Ω, that is, $\|x\| \leq K$, $x \in \Omega$. Choose an integer L and real numbers $y_1 < y_2 < \cdots < y_L$ such that $y_1 = -K$, $y_L = K$ and

$$|y_{i+1} - y_i| < \frac{\epsilon}{2}, \quad i = 1, 2, \ldots, L-1. \tag{3.246}$$

We define a double sequence $v = \{v_{m,n}\}$, $m \geq M$, $n \geq N$ as follows: let $v_{m,n}$ be one of the values $\{y_1, y_2, \dots, y_L\}$ for $M \leq m \leq M_1$, $N \leq n \leq N_1$; let $v_{m,n} = v_{m,N_1}$ for $(m,n) \in D_2'$; let $v_{m,n} = v_{M_1,n}$ for $(m,n) \in D_3'$; let $v_{m,n} = v_{M_1,N_1}$ for $(m,n) \in D_1'$. Clearly, the double sequence $v = \{v_{m,n}\}$, $m \geq M$, $n \geq N$ belongs to X. Let Y be the set of all double sequences v defined as above. Note that Y includes $L^{(M_1-M+1)(N_1-N+1)}$ such double sequences.

We claim that Y is a finite ϵ-net for Ω. For any x in Ω, we must show that Y contains a double sequence v which differs from x by less than ϵ at all positive integer pairs (m,n), $m \geq M$, $n \geq N$. For each $M \leq m \leq M_1$, $N \leq n \leq N_1$, choose $v_{m,n}$ in $\{y_1, y_2, \dots, y_L\}$ such that

$$|x_{m,n} - v_{m,n}| = \min_{1 \leq j \leq L} |x_{m,n} - y_j|. \tag{3.247}$$

Let

$$\begin{aligned} v_{m,n} &= v_{m,N_1}, \quad (m,n) \in D_2', \\ v_{m,n} &= v_{M_1,n}, \quad (m,n) \in D_3', \\ v_{m,n} &= v_{M_1,N_1}, \quad (m,n) \in D_1'. \end{aligned} \tag{3.248}$$

Hence, $v = \{v_{m,n}\}$, $m \geq M$, $n \geq N$ belongs to Y. In view of (3.246) and (3.247), we have

$$|x_{m,n} - v_{m,n}| < \frac{\epsilon}{2}, \quad M \leq m \leq M_1,\ N \leq n \leq N_1. \tag{3.249}$$

For $(m,n) \in D_2'$, (3.241) and (3.249) imply that

$$|x_{m,n} - v_{m,n}| = |x_{m,n} - v_{m,N_1}| \leq |x_{m,n} - x_{m,N_1}| + |x_{m,N_1} - v_{m,N_1}| < \epsilon. \tag{3.250}$$

For $(m,n) \in D_3'$, (3.241) and (3.249) imply that

$$|x_{m,n} - v_{m,n}| = |x_{m,n} - v_{M_1,n}| \leq |x_{m,n} - x_{M_1,n}| + |x_{M_1,n} - v_{M_1,n}| < \epsilon. \tag{3.251}$$

For $(m,n) \in D_1'$, (3.241) and (3.249) imply that

$$\left|x_{m,n} - v_{m,n}\right| = \left|x_{m,n} - v_{M_1,N_1}\right| \leq \left|x_{m,n} - x_{M_1,N_1}\right| + \left|x_{M_1,N_1} - v_{M_1,N_1}\right| < \epsilon. \tag{3.252}$$

Equations (3.249), (3.250), (3.251), and (3.252) imply that $\|v - x\| < \epsilon$. The proof is complete. □

From the above we obtain the following Schauder's fixed point theorem for the difference equations.

Lemma 3.60. *Suppose X is a Banach space and Ω is a closed, bounded, and convex subset of Ω. Suppose T is a continuous mapping such that $T(\Omega)$ is contained in Ω, and suppose that $T(\Omega)$ is uniformly Cauchy. Then T has a fixed point in Ω.*

Theorem 3.61. *Suppose $c \in (0,1]$ and there exist constants $a > 0$, $b > 0$ and $d > 0$ such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c^{-(1/2)(i/k+j/l)} i^{h+a} j^{r+b} F\left(i, j, dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right) < \infty. \tag{3.253}$$

Then (3.240) *has a bounded oscillatory solution $\{x_{m,n}\}$ such that*

$$x_{m,n} = K_1 c^{(1/2)(m/k+n/l)}\left(\cos\frac{2\pi}{k}m\cos\frac{2\pi}{l}n + o(1)\right) \quad \text{as } m,n \longrightarrow \infty, \tag{3.254}$$

where K_1 is some constant.

Proof. By (3.253), we can choose positive integers M, N sufficiently large such that

$$\sum_{i=M}^{\infty}\sum_{j=N}^{\infty} c^{-(1/2)(i/k+j/l)} i^{h+a} j^{r+b} F\left(i, j, dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right) < \frac{d}{3} \tag{3.255}$$

and so that

$$\overline{M} = \min\{M-k, M-\tau\}, \qquad \overline{N} = \min\{N-l, N-\sigma\} \tag{3.256}$$

satisfy

$$\frac{1}{ak}\overline{M}^{-a} < 1, \qquad \frac{1}{bl}\overline{N}^{-b} < 1. \tag{3.257}$$

For every pair of positive integers $p, q \in N$, we define a set $S_{p,q}$ by

$$S_{p,q} = \{(i, j) \mid i, j \in N,\ i \geq p,\ j \geq q\}. \tag{3.258}$$

Let X be the linear space of all bounded real sequences $x = \{x_{m,n}\}$, $m \geq M$, $n \geq N$ endowed with the usual norm

$$\|x\| = \sup_{m\geq M,\, n\geq N} |x_{m,n}|, \tag{3.259}$$

and let

$$\Omega = \left\{x \in X \mid |x_{m,n}| \leq \frac{d}{3}c^{(1/2)(m/k+n/l)}m^{-(1+a)}n^{-(1+b)},\ m \geq M,\ n \geq N\right\} \tag{3.260}$$

with the topology of pointwise convergence, that is, if $\{x^u\}$, $u = 1, 2, \ldots$ is a sequence of elements in Ω, then $\{x^u\}$ converges to x in Ω means that for every $(m, n) \in S_{M,N}$, $\lim_{u\to\infty} x^u_{m,n} = x_{m,n}$. Thus Ω is a closed, bounded, and convex subset.

For every $x \in \Omega$, we associate the function $\overline{x} : S_{M,N} \to R$ defined by

$$\overline{x}_{m,n} = \frac{d}{3}c^{(1/2)(m/k+n/l)} \cos\frac{2\pi}{k}m \cos\frac{2\pi}{l}n - \sum_{i,j=1,i=j}^{\infty} c^{-(1/2)(i+j)}x_{m+ik,n+jl}. \tag{3.261}$$

Notice that

$$\begin{aligned}
&\left|\sum_{i,j=1,i=j}^{\infty} c^{-(1/2)(i+j)}x_{m+ik,n+jl}\right| \\
&\qquad \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c^{-(1/2)(i+j)}\frac{d}{3}c^{(1/2)((m+ik)/k+(n+jl)/l)}(m+ik)^{-(1+a)}(n+jl)^{-(1+b)} \\
&\qquad \leq \frac{d}{3}c^{(1/2)(m/k+n/l)}\frac{m^{-a}}{ak}\frac{n^{-b}}{bl},
\end{aligned} \tag{3.262}$$

for $m \geq \overline{M}$, $n \geq \overline{N}$.

Thus we see that for each $x \in \Omega$, $\overline{x}$ is oscillatory, and moreover,

$$\overline{x}_{m,n} - c\overline{x}_{m-k,n-l} = x_{m,n}, \quad |\overline{x}_{m,n}| \leq dc^{(1/2)(m/k+n/l)}. \tag{3.263}$$

Define an operator $L : \Omega \to \Omega$ by

$$Lx_{m,n} = \begin{cases} \dfrac{(-1)^{h+r+1}}{(h-1)!(r-1)!} \\ \qquad \times \displaystyle\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} (i-m+h-1)^{(h-1)}(j-n+r-1)^{(r-1)} \\ \qquad\qquad \times f(i,j,\overline{x}_{i-\tau,j-\sigma}), & m \geq M,\ n \geq N, \\ Lx_{M,N}, & \text{otherwise}, \end{cases} \tag{3.264}$$

where $n^{(m)}$ denotes the generalized factorial given by $n^{(m)} = n(n-1)\cdots(n-m+1)$.

Claim 1. $L(\Omega) \subset \Omega$. For every $x \in \Omega$ and $m \geq M$, $n \geq N$, we have

$$\begin{aligned} |Lx_{m,n}| &\leq \frac{1}{(h-1)!(r-1)!} \\ &\quad \times \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} (i-m+h-1)^{(h-1)}(j-n+r-1)^{(r-1)} \times |f(i,j,\overline{x}_{i-\tau,j-\sigma})| \\ &\leq \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} i^{h-1} j^{r-1} F(i,j,|\overline{x}_{i-\tau,j-\sigma}|) \\ &\leq \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} i^{h-1} j^{r-1} F(i,j,dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}) \\ &\leq m^{-(1+a)} n^{-(1+b)} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} i^{h+a} j^{r+b} c^{-(1/2)(i/k+j/l)} F(i,j,dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}) \\ &\leq \frac{d}{3} c^{(1/2)(m/k+n/l)} m^{-(1+a)} n^{-(1+b)}. \end{aligned} \tag{3.265}$$

Claim 2. If $\{x^u\}$ converges to x in Ω, then for each pair of integers $s \geq M$, $t \geq N$, we have $\lim_{u\to\infty} \overline{x}^u_{s,t} = \overline{x}_{s,t}$. For every $\varepsilon > 0$, there exist positive integers M_1, N_1 with $M_1 = N_1$ such that

$$\frac{d}{3} \sum_{i,j=M_1+1,\, i=j}^{\infty} (s+ik)^{-(1+a)}(t+jl)^{-(1+b)} < \frac{\varepsilon}{4}. \tag{3.266}$$

For these ε and M_1, N_1, there exists a positive integer N^* such that for every $u \geq N^*$ and every $i \in \{s, s+k, \dots, s+(M_1-1)k, s+M_1k\}$, $j \in \{t, t+l, \dots, t+(N_1-1)l, t+N_1l\}$, we have

$$\left|x_{i,j}^u - x_{i,j}\right| \leq \frac{\varepsilon}{2M_1N_1}c^{(1/2)(M_1+N_1)}. \tag{3.267}$$

Then for every $u \geq N^*$, we see that

$$\begin{aligned}
\left|\overline{x}_{s,t}^u - \overline{x}_{s,t}\right| &= \left|\sum_{i,j=1,\, i=j}^{\infty} c^{-(1/2)(i+j)}\left(x_{s+ik,t+jl}^u - x_{s+ik,t+jl}\right)\right| \\
&\leq \left|\sum_{i,j=1,\, i=j}^{M_1} c^{-(1/2)(i+j)}\left(x_{s+ik,t+jl}^u - x_{s+ik,t+jl}\right)\right| \\
&\quad + \left|\sum_{i,j=M_1+1,\, i=j}^{\infty} c^{-(1/2)(i+j)}\left(x_{s+ik,t+jl}^u - x_{s+ik,t+jl}\right)\right| \\
&\leq \sum_{i,j=1,\, i=j}^{M_1} c^{-(1/2)(M_1+N_1)}\left|x_{s+ik,t+jl}^u - x_{s+ik,t+jl}\right| \\
&\quad + 2\sum_{i,j=M_1+1,\, i=j}^{\infty} \frac{d}{3}c^{(1/2)(s/k+t/l)}(s+ik)^{-(1+a)}(t+jl)^{-(1+b)} < \varepsilon.
\end{aligned} \tag{3.268}$$

Claim 3. L is a continuous operator. Suppose $\{x^u\}$ converges to x in Ω. We will prove that Lx^u converges to Lx in Ω, that is, for every $m \geq M$, $n \geq N$, we claim that $\lim_{u\to\infty} Lx_{m,n}^u = Lx_{m,n}$. Let $\varepsilon > 0$ be given. By (3.245), there exist $M_2 \geq M$, $N_2 \geq N$ such that

$$\begin{gathered}
\sum_{i=M_2+1}^{\infty}\sum_{j=n}^{\infty} i^{h-1}j^{r-1}F\left(i, j, dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right) < \frac{\varepsilon}{6}, \\
\sum_{i=m}^{\infty}\sum_{j=N_2+1}^{\infty} i^{h-1}j^{r-1}F\left(i, j, dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right) < \frac{\varepsilon}{6}.
\end{gathered} \tag{3.269}$$

Since f is continuous on $\{M, M+1, \dots, M_2\} \times \{N, N+1, \dots, N_2\} \times [-d, d]$, f is also uniformly continuous there. From Claim 2, there exists a positive integer N^{**} such that for every $u \geq N^{**}$, and every $i \in \{M, M+1, \dots, M_2-1, M_2\}$, $j \in \{N, N+1, \dots, N_2-1, N_2\}$, we have

$$\left|f\left(i, j, \overline{x}_{i-\tau,j-\sigma}^u\right) - f\left(i, j, \overline{x}_{i-\tau,j-\sigma}\right)\right| < \frac{\varepsilon}{3}\left(\sum_{i=M}^{M_2}\sum_{j=N}^{N_2} i^{h-1}j^{r-1}\right)^{-1}. \tag{3.270}$$

Then, for every $u \geq N^{**}$, we have

$$\begin{aligned} \left| Lx^u_{m,n} - Lx_{m,n} \right| &\leq \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} i^{h-1}j^{r-1}\left| f\left(i,j,\overline{x}^u_{i-\tau,j-\sigma}\right) - f\left(i,j,\overline{x}_{i-\tau,j-\sigma}\right)\right| \\ &\leq \sum_{i=m}^{M_2}\sum_{j=n}^{N_2} i^{h-1}j^{r-1}\left| f\left(i,j,\overline{x}^u_{i-\tau,j-\sigma}\right) - f\left(i,j,\overline{x}_{i-\tau,j-\sigma}\right)\right| \\ &\quad + 2\Bigg(\sum_{i=M_2+1}^{\infty}\sum_{j=n}^{\infty} i^{h-1}j^{r-1}F\left(i,j,dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right) \\ &\qquad + \sum_{i=m}^{\infty}\sum_{j=N_2+1}^{\infty} i^{h-1}j^{r-1}F\left(i,j,dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right)\Bigg) < \varepsilon. \end{aligned} \tag{3.271}$$

Claim 4. $L(\Omega)$ is uniformly Cauchy. The proof is similar to Claim 3, so we omit it here.

Now, by Lemma 3.60, L has a fixed point $w \in \Omega$, that is,

$$\begin{aligned} w_{m,n} &= \frac{(-1)^{h+r+1}}{(h-1)!(r-1)!} \\ &\quad \times \sum_{i=m}^{\infty}\sum_{j=n}^{\infty}(i-m+h-1)^{(h-1)}(j-n+r-1)^{(r-1)} f\left(i,j,\overline{w}_{i-\tau,j-\sigma}\right). \end{aligned} \tag{3.272}$$

Since

$$\overline{w}_{m,n} - c\overline{w}_{m-k,n-l} = w_{m,n}, \tag{3.273}$$

we have

$$\Delta^h_m\Delta^r_n\left(\overline{w}_{m,n} - c\overline{w}_{m-k,n-l}\right) + f\left(m,n,\overline{w}_{m-\tau,n-\sigma}\right) = 0, \tag{3.274}$$

which implies that $\{\overline{w}_{m,n}\}$ is an oscillatory solution of (3.240). From (3.261), $\overline{w}_{m,n}$ satisfies (3.254). The proof is complete. □

Theorem 3.62. *Let $c > 1$. If there exist constants $a > 0$, $b > 0$ and $d > 0$ such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{h+a}j^{r+b}F\left(i,j,dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right) < \infty, \tag{3.275}$$

then (3.240) *has an unbounded oscillatory solution* $\{x_{m,n}\}$ *such that*

$$x_{m,n} = K_2 c^{(1/2)(m/k+n/l)}\left(\cos\frac{2\pi}{k}m\cos\frac{2\pi}{l}n + o(1)\right) \quad \text{as } m,n \longrightarrow \infty, \tag{3.276}$$

where K_2 *is a constant.*

Proof. By the assumption, we can choose positive integers M, N large enough so that

$$\sum_{i=M}^{\infty}\sum_{j=N}^{\infty} i^{h+a} j^{r+b} F\left(i, j, dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right) < \frac{d}{3}c^{(1/2)(m/k+n/l)} \tag{3.277}$$

and so that

$$\overline{M} = \min\{M-k, M-\tau\}, \qquad \overline{N} = \min\{N-l, N-\sigma\} \tag{3.278}$$

satisfy

$$\frac{1}{ak}\overline{M}^{-a} < 1, \qquad \frac{1}{bl}\overline{N}^{-b} < 1. \tag{3.279}$$

Let Ω be defined as in the proof of Theorem 3.61. For each $x \in \Omega$, define $\overline{x}$ by

$$\overline{x}_{m,n} = \frac{d}{3}c^{(1/2)(m/k+n/l)}\cos\frac{2\pi}{k}m\cos\frac{2\pi}{l}n + \sum_{i,j=0,\, i=j}^{\infty} c^{-(1/2)(i+j)}x_{m+ik,n+jl}. \tag{3.280}$$

Thus we can prove that $\overline{x}_{m,n}$ is oscillatory, $\overline{x}_{m,n} - (1/c)\overline{x}_{m+k,n+l} = x_{m,n}$ and $|\overline{x}_{m,n}| < dc^{(1/2)(m/k+n/l)}$.

Next define an operator $L : \Omega \to \Omega$ by

$$Lx_{m,n} = \begin{cases} -\dfrac{(-1)^{h+r+1}}{c(h-1)!(r-1)!} \\ \qquad \times \displaystyle\sum_{i=m+k}^{\infty}\sum_{j=n+l}^{\infty}(i-m-k+h-1)^{(h-1)} \\ \qquad\qquad \times(j-n-l+r-1)^{(r-1)} \\ \qquad\qquad \times f\left(i, j, \overline{x}_{i-\tau,j-\sigma}\right), & m \geq M,\ n \geq N, \\ 0, & \text{otherwise}. \end{cases} \tag{3.281}$$

The remainder of the proof is similar to the proof of Theorem 3.61 and we omit the details here. □

Theorem 3.63. *Suppose $c < 0$ and let $\lambda = -c > 0$. If there exist constants $a > 0$, $b > 0$, and $d > 0$ such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{h+a} j^{r+b} F\left(i, j, d\lambda^{(1/2)((i-\tau)/k+(j-\sigma)/l)}\right) < \infty, \tag{3.282}$$

then (3.240) *has an oscillatory solution $\{x_{m,n}\}$ such that*

$$x_{m,n} = K_3\lambda^{(1/2)(m/k+n/l)}\left(\cos\frac{\pi}{k}m\cos\frac{2\pi}{l}n + o(1)\right) \quad \text{as } m, n \longrightarrow \infty, \tag{3.283}$$

or

$$x_{m,n} = K_4\lambda^{(1/2)(m/k+n/l)}\left(\cos\frac{2\pi}{k}m\cos\frac{\pi}{l}n + o(1)\right) \quad \text{as } m, n \longrightarrow \infty, \tag{3.284}$$

where K_3 and K_4 are constants.

In fact, we define

$$\overline{x}_{m,n} = \frac{d}{3}\lambda^{(1/2)(m/k+n/l)}\cos\frac{\pi}{k}m\cos\frac{2\pi}{l}n - \sum_{i,j=1,\, i=j}^{\infty}(-\lambda)^{-(1/2)(i+j)}x_{m+ik,n+jl}, \tag{3.285}$$

or

$$\overline{x}_{m,n} = \frac{d}{3}\lambda^{(1/2)(m/k+n/l)}\cos\frac{2\pi}{k}m\cos\frac{\pi}{l}n - \sum_{i,j=1,i=j}^{\infty}(-\lambda)^{-(1/2)(i+j)}x_{m+ik,n+jl}, \tag{3.286}$$

thus it satisfies $\overline{x}_{m,n} + \lambda\overline{x}_{m-k,n-l} = x_{m,n}$, define an operator $L : \Omega \to \Omega$ as Theorem 3.61, we can prove Theorem 3.63.

As an application of the above results, we consider the following equation:

$$\Delta_m^h\Delta_n^r\left(x_{m,n} - cx_{m-k,n-l}\right) + p_{m,n}x_{m-\tau,n-\sigma} = q_{m,n}, \tag{3.287}$$

where $p_{m,n}$ and $q_{m,n}$ are positive real double sequences. Let $f(m, n, x_{m-\tau,n-\sigma}) = p_{m,n}x_{m-\tau,n-\sigma} - q_{m,n}$. From Theorems 3.61–3.63, we obtain the following corollaries.

Corollary 3.64. *Suppose $c \in (0, 1]$ and there exist constants $a > 0$, $b > 0$, and $d > 0$ such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c^{-(1/2)(i/k+j/l)} i^{h+a} j^{r+b}\left(\left|p_{i,j}\right| dc^{(1/2)((i-\tau)/k+(j-\sigma)/l)} + \left|q_{i,j}\right|\right) < \infty. \tag{3.288}$$

Then (3.287) *has a bounded oscillatory solution* $\{x_{m,n}\}$ *such that*

$$x_{m,n} = K_1 c^{(1/2)(m/k+n/l)}\left(\cos\frac{2\pi}{k}m\cos\frac{2\pi}{l}n + o(1)\right) \quad \text{as } m,n \longrightarrow \infty, \tag{3.289}$$

where K_1 *is some constant.*

Corollary 3.65. *Let* $c > 1$. *If there exist constants* $a > 0$, $b > 0$, *and* $d > 0$ *such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{h+a} j^{r+b}\left(|p_{i,j}|\, d c^{(1/2)((i-\tau)/k+(j-\sigma)/l)} + |q_{i,j}|\right) < \infty, \tag{3.290}$$

then (3.287) *has an unbounded oscillatory solution* $\{x_{m,n}\}$ *such that*

$$x_{m,n} = K_2 c^{(1/2)(m/k+n/l)}\left(\cos\frac{2\pi}{k}m\cos\frac{2\pi}{l}n + o(1)\right) \quad \text{as } m,n \longrightarrow \infty, \tag{3.291}$$

where K_2 *is a constant.*

Corollary 3.66. *Suppose* $c < 0$ *and let* $\lambda = -c > 0$. *If there exist constants* $a > 0$, $b > 0$, *and* $d > 0$ *such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{h+a} j^{r+b}\left(|p_{i,j}|\, d \lambda^{(1/2)((i-\tau)/k+(j-\sigma)/l)} + |q_{i,j}|\right) < \infty, \tag{3.292}$$

then (3.287) *has an oscillatory solution* $\{x_{m,n}\}$ *such that*

$$x_{m,n} = K_3 \lambda^{(1/2)(m/k+n/l)}\left(\cos\frac{\pi}{k}m\cos\frac{2\pi}{l}n + o(1)\right) \quad \text{as } m,n \longrightarrow \infty, \tag{3.293}$$

or

$$x_{m,n} = K_4 \lambda^{(1/2)(m/k+n/l)}\left(\cos\frac{2\pi}{k}m\cos\frac{\pi}{l}n + o(1)\right) \quad \text{as } m,n \longrightarrow \infty, \tag{3.294}$$

where K_3 *and* K_4 *are constants.*

Example 3.67. Consider the equation

$$\Delta_m^h \Delta_n^r \left(x_{m,n} - x_{m-2,n-2}\right) + \frac{1}{m^\alpha n^\beta}\arctan x_{m-2,m-3} = 0, \qquad m \ge 2,\ n \ge 3, \tag{3.295}$$

where α, β are real numbers.

We only notice that $c = 1$ and $F(m,n,u) = (1/m^\alpha n^\beta)u$, and here take $0 < a < 1$, $0 < b < 1$, and $\alpha \geq h+1$, $\beta \geq r+1$, then (3.253) comes to

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{h+a} j^{r+b} \frac{1}{i^\alpha j^\beta} d < \infty. \tag{3.296}$$

By Theorem 3.61, (3.295) has an oscillatory solution which satisfies (3.254).

3.5. Existence of positive solutions of certain nonlinear PDEs

3.5.1. Existence of positive solutions for the neutral-type equation

We consider nonlinear partial difference equations of the form

$$\Delta_n^h \Delta_m^r \left(x_{m,n} - c x_{m-k,n-l}\right) + (-1)^{h+r+1} p_{m,n} f\left(x_{m-\tau,n-\sigma}\right) = 0. \tag{3.297}$$

With respect to (3.297), throughout we will assume that

(i) $c \in R$, $h,r,k,l \in N_1$, $\tau,\sigma \in N$, $\{p_{m,n}\}_{m=m_0,\,n=n_0}^{\infty,\infty}$ is a double sequence of real numbers,

(ii) $f \in C(R,R)$ is nondecreasing, $xf(x) \geq 0$ for any $x \neq 0$, and $|f(x)| \leq |f(y)|$ as $|x| \leq |y|$.

Let $\delta = \max\{k,\tau\}$, $\eta = \max\{l,\sigma\}$ be fixed nonnegative integers.

Theorem 3.68. *Assume that* $0 \leq c < 1$, $p_{m,n} \geq 0$, *and that there exists a positive double sequence* $\{\lambda_{m,n}\}$ *such that for all sufficiently large* m, n

$$c\frac{\lambda_{m-k,n-l}}{\lambda_{m,n}} + \frac{1}{\lambda_{m,n}}\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}\binom{i-m+r-1}{r-1}\binom{j-n+h-1}{h-1} p_{i,j} f\left(\lambda_{i-\tau,j-\sigma}\right) \leq 1. \tag{3.298}$$

Then (3.297) *has a positive solution* $\{x_{m,n}\}$ *which satisfies* $0 < x_{m,n} \leq \lambda_{m,n}$.

Proof. Let X be the set of all real-bounded double sequences $y = \{y_{m,n}\}$ with the norm $\|y\| = \sup_{m\geq m_0,\, n\geq n_0} |y_{m,n}| < \infty$. Then X is a Banach space. We define a subset Ω of X as

$$\Omega = \left\{y = \{y_{m,n}\} \in X \mid 0 \leq y_{m,n} \leq 1,\ m \geq m_0,\ n \geq n_0\right\}, \tag{3.299}$$

where a partial order on X is defined in the usual way, that is,

$$x, y \in X,\ x \le y \text{ means that } x_{m,n} \le y_{m,n} \text{ for } m \ge m_0,\ n \ge n_0. \tag{3.300}$$

It is easy to see that for any subset S of Ω, there exist $\inf S$ and $\sup S$. We choose $m_1 > m_0,\ n_1 > n_0$ sufficiently large such that (3.298) holds. Set

$$\begin{aligned} D &= N_{m_0} \times N_{n_0}, & D_1 &= N_{m_1} \times N_{n_1}, \\ D_2 &= N_{m_0} \times N_{n_1} \backslash D_1, & D_3 &= N_{m_1} \times N_{n_0} \backslash D_1, \\ D_4 &= D \backslash (D_1 \cup D_2 \cup D_3). \end{aligned} \tag{3.301}$$

Clearly, $D = D_1 \cup D_2 \cup D_3 \cup D_4$. Define a mapping $T : \Omega \to X$ as follows:

$$Ty_{m,n} = \begin{cases} c\dfrac{\lambda_{m-k,n-l}}{\lambda_{m,n}} y_{m-k,n-l} + \dfrac{1}{\lambda_{m,n}} \displaystyle\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \binom{i-m+r-1}{r-1} \\ \qquad \times \dbinom{j-n+h-1}{h-1} p_{i,j} f(\lambda_{i-\tau,j-\sigma} y_{i-\tau,j-\sigma}), & (m,n) \in D_1, \\ \dfrac{n}{n_1} Ty_{m_1,n} + \left(1 - \dfrac{n}{n_1}\right), & (m,n) \in D_2, \\ \dfrac{m}{m_1} Ty_{m,n_1} + \left(1 - \dfrac{m}{m_1}\right), & (m,n) \in D_3, \\ \dfrac{mn}{m_1 n_1} Ty_{m_1,n_1} + \left(1 - \dfrac{mn}{m_1 n_1}\right), & (m,n) \in D_4. \end{cases} \tag{3.302}$$

From (3.302) and noting that $y_{m,n} \le 1$ we have

$$\begin{aligned} 0 \le Ty_{m,n} \le c\frac{\lambda_{m-k,n-l}}{\lambda_{m,n}} &+ \frac{1}{\lambda_{m,n}} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \binom{i-m+r-1}{r-1} \\ &\times \binom{j-n+h-1}{h-1} p_{i,j} f(\lambda_{i-\tau,j-\sigma}) \le 1 \quad \text{for } (m,n) \in D_1, \\ 0 \le Ty_{m,n} \le 1 \quad &\text{for } (m,n) \in D_2 \cup D_3 \cup D_4. \end{aligned} \tag{3.303}$$

Therefore, $T\Omega \subset \Omega$. Clearly, T is nondecreasing. By Knaster's fixed point theorem—Theorem 1.9, there is $y \in \Omega$ such that $Ty = y$, that is,

$$y_{m,n} = \begin{cases} c\dfrac{\lambda_{m-k,n-l}}{\lambda_{m,n}} y_{m-k,n-l} + \dfrac{1}{\lambda_{m,n}} \displaystyle\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \binom{i-m+r-1}{r-1} \\ \qquad \times \dbinom{j-n+h-1}{h-1} p_{i,j} f\left(\lambda_{i-\tau,j-\sigma} y_{i-\tau,j-\sigma}\right), & (m,n) \in D_1, \\ \dfrac{n}{n_1} T y_{m_1,n} + \left(1 - \dfrac{n}{n_1}\right), & (m,n) \in D_2, \\ \dfrac{m}{m_1} T y_{m,n_1} + \left(1 - \dfrac{m}{m_1}\right), & (m,n) \in D_3, \\ \dfrac{mn}{m_1 n_1} T y_{m_1,n_1} + \left(1 - \dfrac{mn}{m_1 n_1}\right), & (m,n) \in D_4. \end{cases} \tag{3.304}$$

It is easy to see that $y_{m,n} > 0$ for $(m,n) \in D_2 \cup D_3 \cup D_4$ and hence $y_{m,n} > 0$ for all $(m,n) \in D_1$. Set

$$x_{m,n} = \lambda_{m,n} y_{m,n}, \tag{3.305}$$

then from (3.304) and (3.305) we have

$$\begin{aligned} x_{m,n} = c x_{m-k,n-l} + \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \binom{i-m+r-1}{r-1} \\ \times \binom{j-n+h-1}{h-1} p_{i,j} f\left(x_{i-\tau,j-\sigma}\right), \quad (m,n) \in D_1, \end{aligned} \tag{3.306}$$

and so

$$\Delta_n^h \Delta_m^r \left(x_{m,n} - c x_{m-k,n-l}\right) + (-1)^{r+h+1} p_{m,n} f\left(x_{m-\tau,n-\sigma}\right) = 0, \quad (m,n) \in D_1, \tag{3.307}$$

which implies that $\{x_{m,n}\}$ is a bounded positive solution of (3.297). The proof of Theorem 3.68 is complete. □

Inequality (3.298) is not easy to verify, but we can derive some explicit sufficient conditions by choosing different $\{\lambda_{m,n}\}$ in (3.298) for the existence of positive solutions of (3.297). For example, by choosing $\lambda_{m,n} = a^{m+n}$ or $\lambda_{m,n} = 1/mn$, respectively, we obtain the following results.

Corollary 3.69. *Assume that* $0 \le c < 1$, $p_{m,n} \ge 0$, *and that there exists a positive number* a *such that for all sufficiently large* m, n

$$ca^{-k-l} + \frac{1}{a^{m+n}} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \binom{i-m+r-1}{r-1} \binom{j-n+h-1}{h-1} p_{i,j} f\left(a^{i+j-\tau-\sigma}\right) \le 1. \tag{3.308}$$

Then (3.297) *has a positive solution* $\{x_{m,n}\}$ *which satisfies* $0 < x_{m,n} \le a^{m+n}$.

Corollary 3.70. *Assume that* $0 \le c < 1$, $p_{m,n} \ge 0$, *and that for all sufficiently large* m, n

$$\begin{aligned} c\frac{mn}{(m-k)(n-l)} &+ mn \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \binom{i-m+r-1}{r-1} \\ &\times \binom{j-n+h-1}{h-1} p_{i,j} f\left(\frac{1}{(i-\tau)(j-\sigma)}\right) \le 1. \end{aligned} \tag{3.309}$$

Then (3.297) *has a positive solution* $\{x_{m,n}\}$ *which satisfies* $0 < x_{m,n} \le 1/mn$.

Theorem 3.71. *Assume that* $c > 1$, $p_{m,n} \le 0$, *and that there exists a positive double sequence* $\{\lambda_{m,n}\}$ *such that for all sufficiently large* m, n

$$\begin{aligned} \frac{\lambda_{m+k,n+l}}{c\lambda_{m,n}} &- \frac{1}{c\lambda_{m,n}} \sum_{i=m+k}^{\infty} \sum_{j=n+l}^{\infty} \binom{i-m-k+r-1}{r-1} \\ &\times \binom{j-n-l+h-1}{h-1} p_{i,j} f\left(\lambda_{i-\tau,j-\sigma}\right) \le 1. \end{aligned} \tag{3.310}$$

Then (3.297) *has a positive solution* $\{x_{m,n}\}$ *which satisfies* $0 < x_{m,n} \le \lambda_{m,n}$.

Proof. Let X and Ω be the sets as in the proof of Theorem 3.68. We define a partial order on X in the usual way. It is easy to see that for any subset S of Ω, there exist inf S and sup S. We choose $m_1 > m_0,\ n_1 > n_0$ sufficiently large such that (3.310) holds. Set

$$\begin{aligned} D &= N_{m_0} \times N_{n_0}, & D_1 &= N_{m_1} \times N_{n_1}, \\ D_2 &= N_{m_0} \times N_{n_1} \backslash D_1, & D_3 &= N_{m_1} \times N_{n_0} \backslash D_1, \\ D_4 &= D \backslash (D_1 \cup D_2 \cup D_3). \end{aligned} \tag{3.311}$$

Clearly, $D = D_1 \cup D_2 \cup D_3 \cup D_4$. Define a mapping $T : \Omega \to X$ as follows:

$$Ty_{m,n} = \begin{cases} \dfrac{c\lambda_{m+k,n+l}}{\lambda_{m,n}} y_{m+k,n+l} \\ \qquad - \dfrac{1}{c\lambda_{m,n}} \displaystyle\sum_{i=m+k}^{\infty} \sum_{j=n+l}^{\infty} \binom{i-m-k+r-1}{r-1} \\ \qquad \times \dbinom{j-n-l+h-1}{h-1} p_{i,j} f(\lambda_{i-\tau,j-\sigma} y_{i-\tau,j-\sigma}), & (m,n) \in D_1, \\ \dfrac{n}{n_1} Ty_{m_1,n} + \left(1 - \dfrac{n}{n_1}\right), & (m,n) \in D_2, \\ \dfrac{m}{m_1} Ty_{m,n_1} + \left(1 - \dfrac{m}{m_1}\right), & (m,n) \in D_3, \\ \dfrac{mn}{m_1 n_1} Ty_{m_1,n_1} + \left(1 - \dfrac{mn}{m_1 n_1}\right), & (m,n) \in D_4. \end{cases} \tag{3.312}$$

From (3.312) and noting that $y_{m,n} \le 1$ and $p_{m,n} \le 0$, we have

$$\begin{aligned} 0 \le Ty_{m,n} &\le \frac{\lambda_{m+k,n+l}}{c\lambda_{m,n}} - \frac{1}{c\lambda_{m,n}} \sum_{i=m+k}^{\infty} \sum_{j=n+l}^{\infty} \binom{i-m-k+r-1}{r-1} \\ &\quad \times \binom{j-n-l+h-1}{h-1} p_{i,j} f(\lambda_{i-\tau,j-\sigma}) \le 1 \quad \text{for } (m,n) \in D_1, \\ 0 \le Ty_{m,n} &\le 1 \quad \text{for } (m,n) \in D_2 \cup D_3 \cup D_4. \end{aligned} \tag{3.313}$$

Therefore, $T\Omega \subset \Omega$. Clearly, T is nondecreasing. By Knaster's fixed point theorem there is $y \in \Omega$ such that $Ty = y$, that is,

$$y_{m,n} = \begin{cases} \dfrac{c\lambda_{m+k,n+l}}{\lambda_{m,n}} y_{m+k,n-l} \\ \qquad -\dfrac{1}{c\lambda_{m,n}} \displaystyle\sum_{i=m+k}^{\infty} \sum_{j=n+l}^{\infty} \binom{i-m-k+r-1}{r-1} \\ \qquad \times \dbinom{j-n-l+h-1}{h-1} p_{i,j} f(\lambda_{i-\tau,j-\sigma} y_{i-\tau,j-\sigma}), & (m,n) \in D_1, \\ \dfrac{n}{n_1} T y_{m_1,n} + \left(1 - \dfrac{n}{n_1}\right), & (m,n) \in D_2, \\ \dfrac{m}{m_1} T y_{m,n_1} + \left(1 - \dfrac{m}{m_1}\right), & (m,n) \in D_3, \\ \dfrac{mn}{m_1 n_1} T y_{m_1,n_1} + \left(1 - \dfrac{mn}{m_1 n_1}\right), & (m,n) \in D_4. \end{cases} \tag{3.314}$$

It is easy to see that $y_{m,n} > 0$ for $(m,n) \in D_2 \cup D_3 \cup D_4$ and hence $y_{m,n} > 0$ for all $(m,n) \in D_1$. Set

$$x_{m,n} = \lambda_{m,n} y_{m,n}, \tag{3.315}$$

then from (3.314) and (3.315) we have

$$\begin{aligned} x_{m,n} = \frac{1}{c} x_{m+k,n+l} - \frac{1}{c} \sum_{i=m+k}^{\infty} \sum_{j=n+l}^{\infty} \binom{i-m-k+r-1}{r-1} \\ \times \binom{j-n-l+h-1}{h-1} p_{i,j} f(x_{i-\tau,j-\sigma}), \quad (m,n) \in D_1, \end{aligned} \tag{3.316}$$

and so

$$\Delta_n^h \Delta_m^r (x_{m,n} - c x_{m-k,n-l}) + (-1)^{r+h+1} p_{m,n} f(x_{m-\tau,n-\sigma}) = 0, \quad (m,n) \in D_1, \tag{3.317}$$

which implies that $\{x_{m,n}\}$ is a bounded positive solution of (3.297). The proof of Theorem 3.71 is complete. □

By choosing $\lambda_{m,n} = a^{m+n}$ and $\lambda_{m,n} = 1/mn$, respectively, we can obtain the following explicit sufficient conditions for the existence of positive solutions.

Corollary 3.72. *Assume that $c > 1$, $p_{m,n} \le 0$, and that there exists a positive number a such that for all sufficiently large m, n*

$$\begin{aligned} \frac{a^{k+l}}{c} - \frac{1}{ca^{m+n}} \sum_{i=m+k}^{\infty} \sum_{j=n+l}^{\infty} \binom{i-m-k+r-1}{r-1} \\ \times \binom{j-n-l+h-1}{h-1} p_{i,j} f\left(a^{i+j-\tau-\sigma}\right) \le 1. \end{aligned} \tag{3.318}$$

Then (3.297) *has a positive solution $\{x_{m,n}\}$ which satisfies $0 < x_{m,n} \le a^{m+n}$.*

Corollary 3.73. *Assume that $c > 1$, $p_{m,n} \le 0$, and that for all sufficiently large m, n*

$$\begin{aligned} \frac{mn}{c(m+k)(n+l)} - \frac{mn}{c} \sum_{i=m+k}^{\infty} \sum_{j=n+l}^{\infty} \binom{i-m-k+r-1}{r-1} \\ \times \binom{j-n-l+h-1}{h-1} p_{i,j} f\left(\frac{1}{(i-\tau)(j-\sigma)}\right) \le 1. \end{aligned} \tag{3.319}$$

Then (3.297) *has a positive solution $\{x_{m,n}\}$ which satisfies $0 < x_{m,n} \le 1/mn$.*

Example 3.74. Consider the partial difference equation

$$\Delta_n \Delta_m x_{m,n} - p_{m,n} x_{m-1,n-1}^{1/3} = 0, \quad m \ge 2,\ n \ge 2, \tag{3.320}$$

where

$$p_{m,n} = \frac{(4mn+2m+2n+1)(m-1)^{2/3}(n-1)^{2/3}}{(m+1)^2(n+1)^2m^2n^2}. \tag{3.321}$$

We take $c = 0$, $h = r = 1$, $f(x) = x^{1/3}$, $\tau = \sigma = 1$ in Corollary 3.70. Obviously, conditions of Corollary 3.70 are satisfied for (3.320). By Corollary 3.70, (3.320) has a positive solution $\{x_{m,n}\}$, which satisfies $0 < x_{m,n} \le 1/mn$. In fact, $\{x_{m,n}\} = 1/m^2n^2$ is such a solution of (3.320).

3.5.2. Existence of nonoscillatory solutions for the neutral-type equation

In this section, we consider the existence of nonoscillatory solutions of the nonlinear partial difference equation of the form

$$\Delta_n^h \Delta_m^r (y_{m,n} + c y_{m-k,n-l}) + F(m, n, y_{m-\tau,n-\sigma}) = 0, \tag{3.322}$$

where $h, r, k, l \in N_1$, $\tau, \sigma \in N_0$, $c \in R$. $F : N_0 \times N_0 \times R \to R$ is continuous.

Theorem 3.75. *Assume that* $c \neq -1$ *and that there exists an interval* $[a, b] \subset R(0 < a < b)$ *such that*

$$\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} (m)^{(r-1)}(n)^{(h-1)} \sup_{w\in[a,b]} \left|F(m, n, w)\right| < \infty. \tag{3.323}$$

Then (3.322) *has a bounded nonoscillatory solution.*

Proof. The proof of this theorem will be divided into five cases in terms of c. Let X be the set of all bounded real double sequence $y = \{y_{m,n}\}$, $m \geq M$, $n \geq N$ with the norm $\|y\| = \sup_{m\geq M,\, n\geq N} |y_{m,n}| < \infty$. X is a Banach space. We define a closed, bounded, and convex subset Ω of X as follows:

$$\Omega = \left\{y = \left\{y_{m,n}\right\} \in X \mid a \leq y_{m,n} \leq b,\ m \geq M,\ n \geq N\right\}. \tag{3.324}$$

Case 1. For the case $-1 < c \leq 0$, choose $m_1 > M$, $n_1 > N$ sufficiently large such that $m_1 - \max\{k, \tau\} \geq M$, $n_1 - \max\{l, \sigma\} \geq N$ and

$$\frac{1}{(r-1)!(h-1)!} \sum_{i=m_1}^{\infty} (i)^{(r-1)} \sum_{j=n_1}^{\infty} (j)^{(h-1)} \sup_{w\in[a,b]} \left|F(i, j, w)\right| \leq \frac{(c+1)(b-a)}{2}. \tag{3.325}$$

Set

$$\begin{aligned}
D &= \left\{(m, n) \mid m \geq M,\ n \geq N\right\}, & D_1 &= \left\{(m, n) \mid m \geq m_1,\ n \geq n_1\right\},\\
D_2 &= \left\{(m, n) \mid M \leq m < m_1,\ n > n_1\right\}, & D_3 &= \left\{(m, n) \mid m > m_1,\ N \leq n < n_1\right\},\\
D_4 &= \left\{(m, n) \mid M \leq m \leq m_1,\ N \leq n \leq n_1\right\}.
\end{aligned} \tag{3.326}$$

Clearly, $D = D_1 \cup D_2 \cup D_3 \cup D_4$.

Define two maps T_1 and $T_2 : \Omega \to X$ by

$$T_1 y_{m,n} = \begin{cases} \dfrac{(c+1)(b+a)}{2} - c y_{m-k,n-l}, & (m,n) \in D_1, \\ T_1 y_{m_1,n}, & (m,n) \in D_2, \\ T_1 y_{m,n_1}, & (m,n) \in D_3, \\ T_1 y_{m_1,n_1}, & (m,n) \in D_4. \end{cases}$$

$$T_2 y_{m,n} = \begin{cases} \dfrac{(-1)^{r+h+1}}{(r-1)!(h-1)!} \displaystyle\sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \\ \qquad \times \displaystyle\sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} F(i,j,y_{i-\tau,j-\sigma}), & (m,n) \in D_1, \\ T_2 y_{m_1,n}, & (m,n) \in D_2, \\ T_2 y_{m,n_1}, & (m,n) \in D_3, \\ T_2 y_{m_1,n_1}, & (m,n) \in D_4. \end{cases} \tag{3.327}$$

(i) We claim that for any $x, y \in \Omega$, $T_1 x + T_2 y \subset \Omega$.
In fact, for every $x, y \in \Omega$ and $m \geq m_1$, $n \geq n_1$, we get

$$\begin{aligned} T_1 x_{m,n} + T_2 y_{m,n} &\leq \frac{(c+1)(b+a)}{2} - cb + \frac{1}{(r-1)!(h-1)!} \\ &\quad \times \sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} \sup_{w \in [a,b]} |F(i,j,w)| \\ &\leq \frac{(c+1)(b+a)}{2} - cb + \frac{(c+1)(b-a)}{2} = b. \end{aligned} \tag{3.328}$$

Furthermore, we have

$$\begin{aligned} T_1 x_{m,n} + T_2 y_{m,n} &\geq \frac{(c+1)(b+a)}{2} - ca - \frac{1}{(r-1)!(h-1)!} \\ &\quad \times \sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \sum_{j=n}^{\infty} (j-n+h-1)^{(h-1)} \sup_{w \in [a,b]} |F(i,j,w)| \\ &\geq \frac{(c+1)(b+a)}{2} - ca - \frac{(c+1)(b-a)}{2} = a. \end{aligned} \tag{3.329}$$

Hence

$$a \leq T_1 x_{m,n} + T_2 y_{m,n} \leq b \quad \text{for } (m,n) \in D. \tag{3.330}$$

Thus we have proved that $T_1 x + T_2 y \subset \Omega$ for any $x, y \in \Omega$.

(ii) We claim that T_1 is a contraction mapping on Ω.

In fact, for $x, y \in \Omega$ and $(m,n) \in D_1$, we have

$$|T_1 x_{m,n} - T_1 y_{m,n}| \leq -c|x_{m-k,n-l} - y_{m-k,n-l}| \leq -c\|x - y\|. \tag{3.331}$$

This implies that

$$||T_1 x - T_1 y|| \leq -c\|x - y\|. \tag{3.332}$$

Since $0 < -c < 1$, we conclude that T_1 is a contraction mapping on Ω.

(iii) We claim that T_2 is completely continuous.

First, we will show that T_2 is continuous. For this, let $y^{(v)} = \{y^{(v)}_{m,n}\} \in \Omega$ be such that $y^{(v)}_{m,n} \to y_{m,n}$ as $v \to \infty$. Because Ω is closed, $y = \{y_{m,n}\} \in \Omega$. For $m \geq m_1$, $n \geq n_1$, we have

$$\begin{aligned} &|T_2 y^{(v)}_{m,n} - T_2 y_{m,n}| \\ &\quad \leq \frac{1}{(r-1)!(h-1)!} \sum_{i=m_1}^{\infty} (i-m+r-1)^{(r-1)} \sum_{j=n_1}^{\infty} (j-n+h-1)^{(h-1)} \\ &\qquad \times |F(i,j,y^{(v)}_{i-\tau,j-\sigma}) - F(i,j,y_{i-\tau,j-\sigma})|. \end{aligned} \tag{3.333}$$

Since

$$\begin{aligned} &(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)} \times |F(i,j,y^{(v)}_{i-\tau,j-\sigma}) - F(i,j,y_{i-\tau,j-\sigma})| \\ &\quad \leq i^{r-1} j^{h-1} (|F(i,j,y^{(v)}_{i-\tau,j-\sigma})| + |F(i,j,y_{i-\tau,j-\sigma})|) \\ &\quad \leq 2 i^{r-1} j^{h-1} \sup_{w \in [a,b]} |F(i,j,w)| \end{aligned} \tag{3.334}$$

and that $|F(i,j,y^{(v)}_{i-\tau,j-\sigma}) - F(i,j,y_{i-\tau,j-\sigma})| \to 0$ as $v \to \infty$, in view of (3.325) and applying the Lebesgue dominated convergence theorem, we conclude that $\lim_{v\to\infty} \|T_2 y^{(v)} - T_{\|} = 0$. This means that T_2 is continuous.

Next, we will show that $T_2\Omega$ is relatively compact. For any given $\varepsilon > 0$, by (3.323), there exist $M_1 \geq m_1$ and $N_1 \geq n_1$ such that

$$\frac{1}{(r-1)!(h-1)!} \sum_{i=M_1}^{\infty} (i+r-1)^{(r-1)} \sum_{j=N_1}^{\infty} (j+h-1)^{(h-1)} \sup_{w \in [a,b]} |F(i,j,w)| < \frac{\varepsilon}{2}. \tag{3.335}$$

By (3.323), we have

$$\sum_{m=m_0}^{\infty}\sum_{n=n_1}^{N_1+1}(m)^{(r-1)}(n)^{(h-1)}\sup_{w\in[a,b]}\left|F(m,n,w)\right|<\infty. \tag{3.336}$$

Hence, there exists an $M'\geq M_1$ such that

$$\frac{1}{(r-1)!(h-1)!}\sum_{i=M'}^{\infty}(i+r-1)^{(r-1)}\sum_{j=n_1}^{N_1+1}(j+h-1)^{(h-1)}\sup_{w\in[a,b]}\left|F(i,j,w)\right|<\varepsilon. \tag{3.337}$$

Similarly, there exists $N'\geq N_1$ such that

$$\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{M_1+1}(i+r-1)^{(r-1)}\sum_{j=N'}^{\infty}(j+h-1)^{(h-1)}\sup_{w\in[a,b]}\left|F(i,j,w)\right|<\varepsilon. \tag{3.338}$$

Then, for any $y=\{y_{m,n}\}\in\Omega$, when $(m,n),(m',n')\in\{(m,n): m\geq M_1,\ n\geq N_1\}$,

$$\begin{aligned}
&\left|T_2y_{m,n}-T_2y_{m',n'}\right| \\
&\quad\leq\frac{1}{(r-1)!(h-1)!}\sum_{i=m}^{\infty}(i-m+r-1)^{(r-1)} \\
&\qquad\times\sum_{i=n}^{\infty}(j-n+h-1)^{(h-1)}\times\left|F\left(i,j,y_{i-\tau,j-\sigma}\right)\right| \\
&\qquad+\frac{1}{(r-1)!(h-1)!}\sum_{i=m'}^{\infty}(i-m+r-1)^{(r-1)} \\
&\qquad\times\sum_{i=n'}^{\infty}(j-n+h-1)^{(h-1)}\times\left|F\left(i,j,y_{i-\tau,j-\sigma}\right)\right| \\
&\quad\leq\frac{1}{(r-1)!(h-1)!}\sum_{i=m}^{\infty}(i-m+r-1)^{(r-1)} \\
&\qquad\times\sum_{i=n}^{\infty}(j-n+h-1)^{(h-1)}\sup_{w\in[a,b]}\left|F(i,j,w)\right|+\frac{1}{(r-1)!(h-1)!} \\
&\qquad\times\sum_{i=m'}^{\infty}(i-m+r-1)^{(r-1)}\sum_{i=n'}^{\infty}(j-n+h-1)^{(h-1)}\sup_{w\in[a,b]}\left|F(i,j,w)\right| \\
&\quad<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned} \tag{3.339}$$

When $(m,n),(m',n') \in \{(m,n) : m \geq M_1,\ n_1 \leq n \leq N_1+1\}$, we have

$$\begin{aligned}
|T_2 y_{m,n} - T_2 y_{m',n'}| &\leq \frac{1}{(r-1)!(h-1)!}\sum_{i=m}^{\infty}(i-m+r-1)^{(r-1)} \\
&\quad \times \sum_{i=n_1}^{N_1}(j-n+h-1)^{(h-1)} \times |F(i,j,y_{i-\tau,j-\sigma})| \\
&\leq \frac{1}{(r-1)!(h-1)!}\sum_{i=M}^{\infty}(i+r-1)^{(r-1)} \\
&\quad \times \sum_{j=n_1}^{N'}(j+h-1)^{(h-1)} \sup_{w\in[a,b]}|F(i,j,w)| < \varepsilon.
\end{aligned} \tag{3.340}$$

Similarly, when $(m,n),(m',n') \in \{(m,n) : m_1 \leq m \leq M_1+1,\ n \geq N_1\}$, we have

$$|T_2 y_{m,n} - T_2 y_{m',n'}| < \varepsilon. \tag{3.341}$$

Let

$$\begin{aligned}
&D_1' = \{(m,n) \mid m > M_1,\ n > N_1\}, \qquad D_2' = \{(m,n) \mid m_1 \leq m \leq M_1,\ n > N_1\}, \\
&D_3' = \{(m,n) \mid m > M_1,\ n_1 \leq n \leq N_1\}.
\end{aligned} \tag{3.342}$$

Then

$$|T_2 y_{m,n} - T_2 y_{m',n'}| < \varepsilon, \quad \text{for } (m,n),(m',n') \in D = D_1' \cup D_2' \cup D_3'. \tag{3.343}$$

This means that $T_2\Omega$ is uniformly Cauchy. Hence, by Lemma 3.59, $T_2\Omega$ is relatively compact. By Theorem 1.14, there is a $y = \{y_{m,n}\} \in \Omega$ such that $T_1 y + T_2 y = y$. Clearly, $y = \{y_{m,n}\}$ is a bounded positive solution of (3.322). This completes the proof in this case.

Case 2. For the case $c < -1$, by (3.323), we choose $m_1 > M$, $n_1 > N$ sufficiently large such that

$$-\frac{1}{c}\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1}^{\infty}(i)^{(r-1)}\sum_{j=n_1}^{\infty}(j)^{(h-1)}\sup_{w\in[a,b]}|F(i,j,w)| \leq \frac{(c+1)(b-a)}{2c}. \tag{3.344}$$

Define two maps mapping T_1 and $T_2 : \Omega \to \Omega$ by

$$T_1 y_{m,n} = \begin{cases} \dfrac{(c+1)(a+b)}{2c} - \dfrac{1}{c} y(m+k, n+l), & (m,n) \in D_1, \\ T_1 y_{m_1,n}, & (m,n) \in D_2, \\ T_1 y_{m,n_1}, & (m,n) \in D_3, \\ T_1 y_{m_1,n_1}, & (m,n) \in D_4. \end{cases}$$

$$T_2 y_{m,n} = \begin{cases} \dfrac{(-1)^{r+h+1}}{c(r-1)!(h-1)!} \displaystyle\sum_{i=m}^{\infty} (i-m+r-1)^{(r-1)} \\ \qquad \times \displaystyle\sum_{j=n+l}^{\infty} (j-n+h-1)^{(h-1)} \\ \qquad \times F(i, j, y_{i-\tau, j-\sigma}), & (m,n) \in D_1, \\ T_2 y_{m_1,n}, & (m,n) \in D_2, \\ T_2 y_{m,n_1}, & (m,n) \in D_3, \\ T_2 y_{m_1,n_1}, & (m,n) \in D_4. \end{cases} \tag{3.345}$$

The rest of the proof is similar to that of Case 1 and it is thus omitted.

Case 3. For the case $0 \le c < 1$, by (3.323), we choose $m_1 > m_0$, $n_1 > n_0$ sufficiently large such that

$$\frac{1}{(r-1)!(h-1)!} \sum_{i=m_1}^{\infty} (i)^{(r-1)} \sum_{j=n_1}^{\infty} (j)^{(h-1)} \sup_{w \in [a,b]} \left| F(i, j, w) \right| \le \frac{(1-c)(b-a)}{2}. \tag{3.346}$$

Define two maps mapping T_1 and $T_2 : \Omega \to \Omega$ as in Case 1, the rest of the proof is similar to that of Case 1 and thus it is omitted.

Case 4. For the case $c > 1$, by (3.323), we choose $m_1 > m_0$, $n_1 > n_0$ sufficiently large such that

$$\frac{1}{c(r-1)!(h-1)!} \sum_{i=m_1}^{\infty} (i)^{(r-1)} \sum_{j=n_1}^{\infty} (j)^{(h-1)} \sup_{w \in [a,b]} \left| F(i, j, w) \right| \le \frac{(c-1)(b-a)}{2c}. \tag{3.347}$$

Define two maps mapping T_1 and $T_2 : \Omega \to \Omega$ as in Case 2, the rest of the proof is similar to that of Case 1 and thus it is omitted.

Case 5. For the case $c = 1$, by (3.323), we choose $m_1 > m_0$, $n_1 > n_0$ sufficiently large such that

$$\frac{1}{(r-1)!(h-1)!}\sum_{i=m_1+k}^{\infty}(i)^{(r-1)}\sum_{j=n_1+l}^{\infty}(j)^{(h-1)}\sup_{w\in[a,b]}\left|F(i,j,w)\right| \le \frac{(b-a)}{2}. \tag{3.348}$$

Define a mapping $T:\Omega\to\Omega$ by

$$Ty_{m,n} = \begin{cases} \dfrac{a+b}{2}+\dfrac{(-1)^{r+h+1}}{(r-1)!(h-1)!} & \\ \quad\times\displaystyle\sum_{u=1}^{\infty}\sum_{i=m+(2u-1)k}^{m+2uk-1}(i-m+r-1)^{(r-1)} & \\ \quad\times\displaystyle\sum_{v=1}^{\infty}\sum_{j=n+(2v-l)l}^{n+2vl-1}(j-n+h-1)^{(n-1)} & \\ \quad\times F\big(i,j,y_{i-\tau,j-\sigma}\big), & (m,n)\in D_1, \\ T_1 y_{m_1,n}, & (m,n)\in D_2, \\ T_1 y_{m,n_1}, & (m,n)\in D_3, \\ T_1 y_{m_1,n_1}, & (m,n)\in D_4. \end{cases} \tag{3.349}$$

Proceeding similarly as in the proof of Case 1, we obtain $T\Omega\subset\Omega$ and the mapping T is completely continuous. By Lemma 3.60, there is a $y\in\Omega$ such that $Ty = y$, therefore for $(m,n)\in D_1$,

$$\begin{aligned} y_{m,n}+y_{m-k,n-l} = a+b+\frac{(-1)^{r+h+1}}{(r-1)!(h-1)!}&\sum_{i=m}^{\infty}(i-m+r-1)^{(n-1)} \\ \times&\sum_{j=n}^{\infty}(j-n+h-1)^{(n-1)}F\big(i,j,y_{i-\tau,j-\sigma}\big). \end{aligned} \tag{3.350}$$

Clearly, $y=\{y_{m,n}\}$ is a bounded positive solution of (3.322). This completes the proof of Theorem 3.75. □

Theorem 3.76. *Assume that* $c=-1$ *and that there exists an interval* $[a,b]\subset\mathbf{R}$ $(0<a<b)$ *such that*

$$\sum_{m=M}^{\infty}\sum_{n=N}^{\infty}mn(m)^{(r-1)}(n)^{(n-1)}\sup_{w\in[a,b]}\left|F(m,n,w)\right|<\infty. \tag{3.351}$$

Then (3.322) *has a bounded nonoscillatory solution.*

Proof. By the known result, (3.351) is equivalent to

$$\sum_{u=0}^{\infty}\sum_{m=M+uk}^{\infty}(m)^{(r-1)}\sum_{v=0}^{\infty}\sum_{n=N+vl}^{\infty}(n)^{(h-1)}\sup_{w\in[a,b]}\left|F(m,n,w)\right|<\infty. \tag{3.352}$$

We choose $m_1 > M$, $n_1 > N$ sufficiently large such that

$$\frac{1}{(r-1)!(h-1)!}\sum_{u=0}^{\infty}\sum_{i=m_1+uk}^{\infty}(i)^{(r-1)}\sum_{v=0}^{\infty}\sum_{j=n_1+vl}^{\infty}(j)^{(h-1)}\sup_{w\in[a,b]}\left|F(i,j,w)\right|\le\frac{(b-a)}{2}. \tag{3.353}$$

We define

$$\Omega=\left\{y=\left\{y_{m,n}\right\}\in X \mid a\le y_{m,n}\le b,\ (m,n)\in D\right\}. \tag{3.354}$$

Define a mapping $T:\Omega\to\Omega$ by

$$Ty_{m,n}=\begin{cases}\dfrac{a+b}{2}+\dfrac{(-1)^{r+h}}{(r-1)!(h-1)!}\displaystyle\sum_{u=1}^{\infty}\sum_{i=m+uk}^{\infty}(i-m+r-1)^{(r-1)} & \\ \qquad\times\displaystyle\sum_{v=1}^{\infty}\sum_{j=n_1+vl}^{\infty}(j-n+h-1)^{(n-1)}F\left(i,j,y_{i-\tau,j-\sigma}\right), & (m,n)\in D_1,\\ T_1y_{m_1,n}, & (m,n)\in D_2,\\ T_1y_{m,n_1}, & (m,n)\in D_3,\\ T_1y_{m_1,n_1}, & (m,n)\in D_4.\end{cases} \tag{3.355}$$

Proceeding similarly as in the proof of Theorem 3.75, we obtain $T\Omega\subset\Omega$ and the mapping T is completely continuous. By Lemma 3.60, there is a $y\in\Omega$ such that $Ty=y$, therefore for $(m,n)\in D_1$,

$$\begin{aligned}&y_{m,n}-y_{m-k,n-l}\\&\quad=\frac{(-1)^{r+h+1}}{(r-1)!(h-1)!}\sum_{i=m}^{\infty}(i-m+r-1)^{(r-1)}\sum_{j=n}^{\infty}(j-n+h-1)^{(n-1)}F\left(i,j,y_{i-\tau,j-\sigma}\right).\end{aligned} \tag{3.356}$$

Clearly, $y=\{y_{m,n}\}$ is a bounded positive solution of (3.322). This completes the proof of Theorem 3.76. □

Example 3.77. Consider the nonlinear partial difference equation

$$\Delta_n^h\Delta_m^r\left(y_{m,n}+cy_{m-k,n-l}\right)+\frac{1}{m^{\alpha}n^{\beta}}y_{m-\tau,n-\sigma}^{\theta}=0, \tag{3.357}$$

where r, h, k, l, τ, σ, and θ are positive integers, $c\neq -1$, $\alpha,\beta\in R^{+}$.

If $\alpha > r$, $\beta > h$, for any real number $b > a > 0$,

$$\begin{aligned}&\sum_{m=M}^{\infty}\sum_{n=N}^{\infty}(m)^{(r-1)}(n)^{(h-1)}\sup_{w\in[a,b]}\left\{\frac{w^{\theta}}{m^{\alpha}n^{\beta}}\right\}\\&\quad\le\sum_{m=M}^{\infty}\sum_{n=N}^{\infty}m^{r-1}n^{h-1}\frac{b^{\theta}}{m^{\alpha}n^{\beta}}=b^{\theta}\sum_{m=M}^{\infty}\frac{1}{m^{\alpha+1-r}}\sum_{n=N}^{\infty}\frac{1}{n^{\beta+1-n}}<\infty.\end{aligned}\tag{3.358}$$

By Theorem 3.75, (3.357) has a bounded positive solution.

If $\alpha > r+1$, $\beta > h+1$, for any real number $b > a > 0$,

$$\begin{aligned}&\sum_{m=M}^{\infty}\sum_{n=N}^{\infty}mn(m)^{(r-1)}(n)^{(h-1)}\sup_{w\in[a,b]}\left\{\frac{w^{\theta}}{m^{\alpha}n^{\beta}}\right\}\\&\quad\le\sum_{m=M}^{\infty}\sum_{n=N}^{\infty}m^{r}n^{h}\frac{b^{\theta}}{m^{\alpha}n^{\beta}}=b^{\theta}\sum_{m=m_0}^{\infty}\frac{1}{m^{\alpha-r}}\sum_{n=n_0}^{\infty}\frac{1}{n^{\beta-n}}<\infty.\end{aligned}\tag{3.359}$$

By Theorem 3.76, (3.357) has a bounded positive solution.

In the following, we present some results for the existence of unbounded positive solutions of the nonlinear partial difference equation

$$\Delta_m^h\Delta_n^r\left(x_{m,n}-c_{m,n}x_{m-k,n-l}\right)+f\left(m,n,x_{m-\tau,n-\sigma}\right)=0,\tag{3.360}$$

where $m,n\in N_1$, $c_{m,n}\ge 0$, $m\ge m_0$, $n\ge n_0$; $h,r,k,l\in N_1$; $\tau,\sigma\in N_0$. $f(m,n,u)$ is of one sign on $N_{m_0}\times N_{n_0}\times(0,\infty)$ and $|f(m,n,u)|$ is nondecreasing in u for $u\in(0,\infty)$ and $(m,n)\in N_{m_0}\times N_{n_0}$.

In the following, we note that

$$R_i(m,n)=\left[\left(N_{m+ik}\setminus N_{m+(i+1)k}\right)\times N_{n+il}\right]\cup\left[N_{m+ik}\times\left(N_{n+il}\setminus N_{n+(i+1)l}\right)\right],\quad i\in Z.\tag{3.361}$$

Theorem 3.78. *Assume that* λ, μ *are integers with* $0\le\lambda\le h-1$, $0\le\mu\le r-1$ *and that there exists a positive sequence* $\{u_{m,n}^{\lambda\mu}\}$ *defined on* $N_{m_0-k}\times N_{n_0-l}$ *such that*

$$0<\liminf_{m,n\to\infty}\frac{u_{m,n}^{\lambda\mu}-c_{m,n}u_{m-k,n-l}^{\lambda\mu}}{m^{\lambda}n^{\mu}}\le\limsup_{m,n\to\infty}\frac{u_{m,n}^{\lambda\mu}-c_{m,n}u_{m-k,n-l}^{\lambda\mu}}{m^{\lambda}n^{\mu}}<\infty.\tag{3.362}$$

Then (3.360) *has a positive solution* $\{x_{m,n}\}$ *such that*

$$\frac{x_{m,n}-c_{m,n}x_{m-k,n-l}}{m^{\lambda}n^{\mu}}\longrightarrow\text{const}>0\quad\text{as } m,n\longrightarrow\infty\tag{3.363}$$

if and only if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}i^{h-\lambda-1}j^{r-\mu-1}\left|f\left(i,j,au_{i-\tau,j-\sigma}^{\lambda\mu}\right)\right|<\infty\quad\text{for some } a>0.\tag{3.364}$$

Moreover, if $\{x_{m,n}\}$ *is a solution of* (3.360) *satisfying* (3.363), *then*

$$c_* u_{m,n}^{\lambda\mu} \le x_{m,n} \le c^* u_{m,n}^{\lambda\mu} \quad \text{for all large } m, n, \tag{3.365}$$

where c_*, c^* *are positive constants.*

To prove this result, we will prove the following lemmas first.

Lemma 3.79. *Assume that* $u, v : N_{m_0-k} \times N_{n_0-l} \to R$ *satisfy*

$$\begin{gathered} u_{m,n} - c_{m,n}u_{m-k,n-l} \ge v_{m,n} - c_{m,n}v_{m-k,n-l}, \quad m \ge m_0,\ n \ge n_0, \\ u_{m,n} \ge v_{m,n}, \quad (m,n) \in R_{-1}(m_0, n_0). \end{gathered} \tag{3.366}$$

Then

$$u_{m,n} \ge v_{m,n}, \quad m \ge m_0 - k,\ n \ge n_0 - l. \tag{3.367}$$

Proof. It is easy to see $N_{m_0-k} \times N_{n_0-l} = \bigcup_{i=-1}^{\infty} R_i(m_0, n_0)$. By the assumption, $u_{m,n} \ge v_{m,n}$ for $(m,n) \in R_{-1}(m_0, n_0)$.

Assume that $u_{m,n} \ge v_{m,n}$, $(m,n) \in R_i(m_0, n_0)$ for some $i = -1, 0, 1, 2, \ldots$. Then

$$u_{m,n} \ge v_{m,n} - c_{m,n}v_{m-k,n-l} + c_{m,n}u_{m-k,n-l} \ge v_{m,n} \quad \text{for } (m,n) \in R_{i+1}(m_0, n_0). \tag{3.368}$$

By induction, the proof is complete. □

Next, consider the initial value problem(IVP)

$$(I) \begin{cases} u_{m,n} - c_{m,n}u_{m-k,n-l} = c, & m \ge m_0,\ n \ge n_0, \\ u_{m,n} = c\varphi_{m,n}, & (m,n) \in R_{-1}(m_0, n_0), \end{cases} \tag{3.369}$$

where $c \in R$, $\varphi : R_{-1}(m_0, n_0) \to R$ satisfies

$$\begin{gathered} \varphi_{m_0,n_0} - c_{m_0,n_0}\varphi_{m_0-k,n_0-l} = 1, \\ \inf_{(m,n)\in R_{-1}(m_0,n_0)} \{\varphi_{m,n}\} > 0. \end{gathered} \tag{3.370}$$

By the method of steps, we can show that IVP(I) has a unique solution u on $N_{m_0-k} \times N_{n_0-l}$. We denote the solution of IVP(I) by $u_\varphi(m,n,c)$. By the uniqueness of solutions of IVP(I), it is clear that, for any $\gamma \in R$,

$$\gamma u_\varphi(m,n,c) = u_\varphi(m,n,\gamma c), \quad m \ge m_0 - k,\ n \ge n_0 - l. \tag{3.371}$$

From Lemma 3.79, we obtain the following lemma.

Lemma 3.80. *Let $u_\varphi(m,n,c)$ be a solution of (I) and suppose that $v : N_{m_0-k} \times N_{n_0-l} \to R$ satisfies*

$$v_{m,n} - c_{m,n}v_{m-k,n-l} \le c, \quad m \ge m_0,\ n \ge n_0,$$
$$v_{m,n} \le c\varphi_{m,n}, \quad (m,n) \in R_{-1}(m_0,n_0). \tag{3.372}$$

Then

$$v_{m,n} \le u_\varphi(m,n,c), \quad m \ge m_0 - k,\ n \ge n_0 - l. \tag{3.373}$$

Lemma 3.81. *Suppose that y, ψ, $\{y^i\}$, $\{\psi^i\}$, $i \in N$ satisfy the following conditions:*

(i) *y^i, $y : N_{m_0} \times N_{n_0} \to R$, and y^i converges to y as $i \to \infty$ for any $(m,n) \in N_{m_0} \times N_{n_0}$;*

(ii) *ψ^i, $\psi : R_{-1}(m_0,n_0) \to R$, and ψ^i converges to ψ as $i \to \infty$ for any $(m,n) \in R_{-1}(m_0,n_0)$.*

Let x^i, $i = 1,2,\dots$ and $x : N_{m_0-k} \times N_{n_0-l} \to R$ be solutions of

$$x^i_{m,n} - c_{m,n}x^i_{m-k,n-l} = y^i_{m,n}, \quad m \ge m_0,\ n \ge n_0,$$
$$x^i_{m,n} = \psi^i_{m,n}, \quad (m,n) \in R_{-1}(m_0,n_0),$$
$$x_{m,n} - c_{m,n}x_{m-k,n-l} = y_{m,n}, \quad m \ge m_0,\ n \ge n_0,$$
$$x_{m,n} = \psi_{m,n}, \quad (m,n) \in R_{-1}(m_0,n_0), \tag{3.374}$$

respectively. Then x^i converges to x as $i \to \infty$ for any $(m,n) \in N_{m_0-k} \times N_{n_0-l}$.

Proof. Take a positive constant c and a positive function φ on $R_{-1}(m_0,n_0)$ satisfying (3.370) and consider the initial value problem (I). We note that

$$u_\varphi(m,n,c) > 0, \quad m \ge m_0 - k,\ n \ge n_0 - l. \tag{3.375}$$

For any $\varepsilon > 0$, there exists $i_0 \in N$ such that if $i \ge i_0$ then

$$|y^i_{m,n} - y_{m,n}| \le \varepsilon, \quad m \ge m_0,\ n \ge n_0,$$
$$|\psi^i_{m,n} - \psi_{m,n}| \le \varepsilon\varphi_{m,n}, \quad (m,n) \in R_{-1}(m_0,n_0). \tag{3.376}$$

Hence we have

$$(x^i_{m,n} - x_{m,n}) - c_{m,n}(x^i_{m-k,n-l} - x_{m-k,n-l}) \le \varepsilon, \quad m \ge m_0,\ n \ge n_0,$$
$$x^i_{m,n} - x_{m,n} \le \varepsilon\varphi_{m,n}, \quad (m,n) \in R_{-1}(m_0,n_0). \tag{3.377}$$

From Lemma 3.80 and (3.371), we obtain

$$x^i_{m,n} - x_{m,n} \le u_\varphi(m,n,\varepsilon) = \varepsilon u_\varphi(m,n,1) \quad \text{for } m \ge m_0 - k,\ n \ge n_0 - l. \tag{3.378}$$

Similarly, we have

$$x_{m,n} - x^i_{m,n} \le u_\varphi(m,n,\varepsilon) = \varepsilon u_\varphi(m,n,1) \quad \text{for } m \ge m_0 - k,\ n \ge n_0 - l. \tag{3.379}$$

Thus we obtain

$$\left| x^i_{m,n} - x_{m,n} \right| \le u_\varphi(m,n,\varepsilon) = \varepsilon u_\varphi(m,n,1) \quad \text{for } m \ge m_0 - k,\ n \ge n_0 - l, \tag{3.380}$$

which implies the conclusion of Lemma 3.81. □

Lemma 3.82. *Let λ, μ be integers with $0 \le \lambda \le h-1$, $0 \le \mu \le r-1$, and let $v^{\lambda\mu}$ be a positive function on $N_{m_0-k} \times N_{n_0-l}$ satisfying*

$$\frac{v^{\lambda\mu}_{m,n} - c_{m,n} v^{\lambda\mu}_{m-k,n-l}}{m^\lambda n^\mu} = 1 \quad \text{for } m \ge m_0,\ n \ge n_0. \tag{3.381}$$

Then, for any positive function $u^{\lambda\mu}$ satisfying (3.362), *there exist $M \ge m_0$, $N \ge n_0$ and positive constants c_*, c^* such that*

$$c_* v^{\lambda\mu}_{m,n} \le u^{\lambda\mu}_{m,n} \le c^* v^{\lambda\mu}_{m,n} \quad \text{for } m \ge M - k,\ n \ge N - l. \tag{3.382}$$

Proof. We can choose sufficiently large $M \ge m_0$, $N \ge n_0$, a sufficiently small $c_* > 0$, and a sufficiently large $c^* > 0$ such that

$$\begin{gathered} c_* \le \frac{u^{\lambda\mu}_{m,n}}{m^\lambda n^\mu} - c_{m,n}\frac{(m-k)^\lambda (n-l)^\mu}{m^\lambda n^\mu} \cdot \frac{u^{\lambda\mu}_{m-k,n-l}}{(m-k)^\lambda (n-l)^\mu} \le c^*, \quad m \ge M,\ n \ge N, \\ \frac{c_* v^{\lambda\mu}_{m,n}}{m^\lambda n^\mu} \le \frac{u^{\lambda\mu}_{m,n}}{m^\lambda n^\mu} \le \frac{c^* v^{\lambda\mu}_{m,n}}{m^\lambda n^\mu}, \quad (m,n) \in R_{-1}(M,N). \end{gathered} \tag{3.383}$$

Since by (3.371) and (3.362), we can obtain that there exist two constants d_* and d^* such that

$$d_* \le \frac{u^{\lambda\mu}_{m,n} - c_{m,n} u^{\lambda\mu}_{m-k,n-l}}{v^{\lambda\mu}_{m,n} - c_{m,n} v^{\lambda\mu}_{m-k,n-l}} \le d^*. \tag{3.384}$$

Because $u^{\lambda\mu}_{m,n}$ and $v^{\lambda\mu}_{m,n}$ are infinity, as $m, n \to \infty$, we can obtain that the order of the infinity $u^{\lambda\mu}_{m,n}$ and $v^{\lambda\mu}_{m,n}$ are the same which implies that the latter inequality holds.

Next, we have

$$\frac{c^* v^{\lambda\mu}_{m,n}}{m^\lambda n^\mu} - c_{m,n}\frac{(m-k)^\lambda (n-l)^\mu}{m^\lambda n^\mu} \cdot \frac{c^* v^{\lambda\mu}_{m-k,n-l}}{(m-k)^\lambda (n-l)^\mu} = c^*, \quad m \ge M,\ n \ge N. \tag{3.385}$$

Applying Lemma 3.80 with c, $\varphi_{m,n}$, $c_{m,n}$ replaced by c^*, $\overset{\lambda\mu}{v}_{m,n}/m^\lambda n^\mu$ and $c_{m,n}((m-k)^\lambda(n-l)^\mu/m^\lambda n^\mu)$, respectively, we obtain

$$\frac{\overset{\lambda\mu}{u}_{m,n}}{m^\lambda n^\mu} \le \frac{c^* \overset{\lambda\mu}{v}_{m,n}}{m^\lambda n^\mu}, \quad m \ge M-k,\ n \ge N-l. \tag{3.386}$$

In the same way, we get

$$\frac{\overset{\lambda\mu}{u}_{m,n}}{m^\lambda n^\mu} \ge \frac{c_* \overset{\lambda\mu}{v}_{m,n}}{m^\lambda n^\mu}, \quad m \ge M-k,\ n \ge N-l. \tag{3.387}$$

The proof is complete. □

*P*roof of Theorem 3.78. Let $\{\overset{\lambda\mu}{v}_{m,n}\}$ be a positive function satisfying (3.381). For the function $\{\overset{\lambda\mu}{u}_{m,n}\}$ in the statement of the theorem, we have (3.382). Assume that x is a positive solution of (3.360) on $N_M \times N_N$ satisfying (3.363). Applying Lemma 3.82 to the case of $u^{\lambda\mu} = x$, we also have

$$c_1 \overset{\lambda\mu}{v}_{m,n} \le x_{m,n} \le c_2 \overset{\lambda\mu}{v}_{m,n} \quad \text{for all large } m, n, \tag{3.388}$$

where c_1, c_2 are positive constants. From (3.382) and (3.388), we obtain

$$\frac{c_1}{c^*} \overset{\lambda\mu}{u}_{m,n} \le x_{m,n} \le \frac{c_2}{c_*} \overset{\lambda\mu}{u}_{m,n} \quad \text{for all large } m, n, \tag{3.389}$$

which implies (3.365).

By (3.363), we have

$$\lim_{m,n\to\infty} \Delta_m^i \Delta_n^j (x_{m,n} - c_{m,n} x_{m-k,n-l}) = 0, \tag{3.390}$$

$(i, j) \in [(N_{\lambda+1} \setminus N_{h+1}) \times (N_0 \setminus N_{r+1})] \cup [(N_0 \setminus N_h) \times (N_{\mu+1} \setminus N_{r+1})] \setminus (h, r)$ and

$$\lim_{m,n\to\infty} \Delta_m^\lambda \Delta_n^\mu (x_{m,n} - c_{m,n} x_{m-k,n-l}) = \text{const} > 0. \tag{3.391}$$

Then sum (3.360) repeatedly, we have

$$\begin{aligned} &\Delta_m^\lambda \Delta_n^\mu (x_{m,n} - c_{m,n} x_{m-k,n-l}) \\ &\quad = \text{const} + (-1)^{h+r-\lambda-\mu-1} \frac{1}{(h-\lambda-1)!(r-\mu-1)!} \\ &\qquad \times \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} (i-m+h-\lambda-1)^{(h-\lambda-1)} (j-n+r-\mu-1)^{(r-\mu-1)} f(i, j, x_{i-\tau, j-\sigma}). \end{aligned} \tag{3.392}$$

for $m \geq M$, $n \geq N$. By (3.389), we have

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} i^{h-\lambda-1} j^{r-\mu-1} \left| f\left(i, j, au_{i-\tau,j-\sigma}^{\lambda\mu}\right) \right| < \infty, \quad m \geq M,\ n \geq N. \tag{3.393}$$

Conversely, we assume that (3.364) holds. By virtue of (3.382), we may assume that

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} i^{h-\lambda-1} j^{r-\mu-1} \left| f\left(i, j, av_{i-\tau,j-\sigma}^{\lambda\mu}\right) \right| < \infty. \tag{3.394}$$

Therefore we can choose $M \geq m_0$, $N \geq n_0$ so large such that

$$\begin{gathered} \widetilde{M} = \min\{M-k, M-\tau\} \geq m, \qquad \widetilde{N} = \min\{N-l, N-\sigma\} \geq n, \\ \sum_{i=M}^{\infty}\sum_{j=N}^{\infty} i^{h-\lambda-1} j^{r-\mu-1} \left| f\left(i, j, av_{i-\tau,j-\sigma}^{\lambda\mu}\right) \right| < \frac{1}{4} a\lambda!\mu!(h-\lambda-1)!(r-\mu-1)!. \end{gathered} \tag{3.395}$$

Let X denote all functions $\{y_{m,n}\}$ defined on $N_M \times N_N$ with $\sup_{m\geq M, n\geq N} |y_{m,n}|/(m^\lambda n^\mu) < \infty$. Define the subset Ω in X by

$$\Omega = \left\{ y \in X \mid \frac{1}{2} a m^\lambda n^\mu \leq y_{m,n} \leq a m^\lambda n^\mu,\ m \geq M,\ n \geq N \right\}. \tag{3.396}$$

Clearly Ω is a nonempty, closed, and convex subset of X.

For $y \in \Omega$, let x be a solution of the following equation:

$$\begin{aligned} x_{m,n} - c_{m,n} x_{m-k,n-l} &= y_{m,n}, \quad m \geq M,\ n \geq N, \\ x_{m,n} &= \frac{y_{M,N}}{M^\lambda N^\mu} v_{m,n}^{\lambda\mu}, \quad (m,n) \in R', \end{aligned} \tag{3.397}$$

where $R' = [(N_{\widetilde{M}} \setminus N_{M+1}) \times N_{\widetilde{N}}] \cup [N_{\widetilde{M}} \times (N_{\widetilde{N}} \setminus N_N)]$. By the method of steps we see that x is uniquely determined as a positive function on $N_{\widetilde{M}} \times N_{\widetilde{N}}$. We define the operator S by

$$\begin{aligned} Sy_{m,n} = {} & \frac{3}{4} a + (-1)^{h+r-1} \frac{1}{(h-1)!(r-1)!} \\ & \times \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} (i-m+h-1)^{(h-1)} (j-n+r-1)^{(r-1)} f\left(i, j, x_{i-\tau,j-\sigma}\right); \end{aligned} \tag{3.398}$$

for the case of $\lambda = \mu = 0$ and

$$Sy_{m,n} = \frac{3}{4}am^{\lambda}n^{\mu} + (-1)^{h+r-\lambda-\mu-1}\sum_{s=M}^{m-1}\sum_{t=N}^{n-1}\frac{(m-s)^{(\lambda-1)}(n-t)^{(\mu-1)}}{(\lambda-1)!(\mu-1)!} \times \sum_{i=s}^{\infty}\sum_{j=t}^{\infty}\frac{(i-s+h-\lambda-1)^{(h-\lambda-1)}(j-t+r-\mu-1)^{(r-\mu-1)}}{(h-\lambda-1)!(r-\mu-1)!} \times f(i,j,x_{i-\tau,j-\sigma}), \tag{3.399}$$

for the case of at least one of λ and $\mu \neq 0$. Here we assume that the general factor $n^{(m)} = 1$ for $m \leq 0$. It is obvious that we can draw the conclusion if we can prove that S has a fixed point in Ω.

First we show that Sy is well defined on $N_{\widetilde{M}} \times N_{\tilde{N}}$ for each $y \in \Omega$ and that $S\Omega \subset \Omega$. Let $y \in \Omega$. We see that

$$\begin{gathered}\frac{x_{m,n}}{m^{\lambda}n^{\mu}} - c_{m,n}\frac{(m-k)^{\lambda}(n-l)^{\mu}}{m^{\lambda}n^{\mu}} \cdot \frac{x_{m-k,n-l}}{(m-k)^{\lambda}(n-l)^{\mu}} = \frac{y_{m,n}}{m^{\lambda}n^{\mu}} \leq a, \quad m \geq M,\ n \geq N, \\ \frac{x_{m,n}}{m^{\lambda}n^{\mu}} = \frac{y_{M,N}}{M^{\lambda}N^{\mu}} \cdot \frac{\overset{\lambda\mu}{v}_{m,n}}{m^{\lambda}n^{\mu}} \leq a\frac{\overset{\lambda\mu}{v}_{m,n}}{m^{\lambda}n^{\mu}}, \quad (m,n) \in R'.\end{gathered} \tag{3.400}$$

On the other hand, for $m \geq M$, $n \geq N$, $a\overset{\lambda\mu}{v}_{m,n}/m^{\lambda}n^{\mu}$ satisfies

$$a\frac{\overset{\lambda\mu}{v}_{m,n}}{m^{\lambda}n^{\mu}} - c_{m,n}\frac{(m-k)^{\lambda}(n-l)^{\mu}}{m^{\lambda}n^{\mu}} \cdot \frac{a\overset{\lambda\mu}{v}_{m-k,n-l}}{(m-k)^{\lambda}(n-l)^{\mu}} = a. \tag{3.401}$$

According to Lemma 3.79, we obtain

$$\frac{x_{m,n}}{m^{\lambda}n^{\mu}} \leq a\frac{\overset{\lambda\mu}{v}_{m,n}}{m^{\lambda}n^{\mu}}, \quad m \geq \widetilde{M},\ n \geq \tilde{N}. \tag{3.402}$$

Hence,

$$x_{m,n} \leq a\overset{\lambda\mu}{v}_{m,n}, \quad m \geq \widetilde{M},\ n \geq \tilde{N}. \tag{3.403}$$

In a similar way, we obtain

$$x_{m,n} \geq \frac{1}{2}a\overset{\lambda\mu}{v}_{m,n}, \quad m \geq \widetilde{M},\ n \geq \tilde{N}. \tag{3.404}$$

Thus, for $0 \le \lambda \le h-1$, $0 \le \mu \le r-1$, it follows that

$$
\begin{aligned}
Sy_{m,n} \le{}& \frac{3}{4}am^{\lambda}n^{\mu} + \sum_{s=M}^{m-1}\sum_{t=N}^{n-1}\frac{(m-s)^{(\lambda-1)}(n-t)^{(\mu-1)}}{(\lambda-1)!(\mu-1)!} \\
&\times \sum_{i=s}^{\infty}\sum_{j=t}^{\infty}\frac{(i-s+h-\lambda-1)^{(h-\lambda-1)}(j-t+r-\mu-1)^{(r-\mu-1)}}{(h-\lambda-1)!(r-\mu-1)!} \\
&\qquad \times \left| f\left(i,j,av^{\lambda\mu}_{i-\tau,j-\sigma}\right)\right| \\
\le{}& \frac{3}{4}am^{\lambda}n^{\mu} + \frac{1}{4}am^{\lambda}n^{\mu} = am^{\lambda}n^{\mu}
\end{aligned}
\tag{3.405}
$$

for $m \ge M$, $n \ge N$, and

$$
\begin{aligned}
Sy_{m,n} \ge{}& \frac{3}{4}am^{\lambda}n^{\mu} - \sum_{s=M}^{m-1}\sum_{t=N}^{n-1}\frac{(m-s)^{(\lambda-1)}(n-t)^{(\mu-1)}}{(\lambda-1)!(\mu-1)!} \\
&\times \sum_{i=s}^{\infty}\sum_{j=t}^{\infty}\frac{(i-s+h-\lambda-1)^{(h-\lambda-1)}(j-t+r-\mu-1)^{(r-\mu-1)}}{(h-\lambda-1)!(r-\mu-1)!} \\
&\qquad \times \left| f\left(i,j,av^{\lambda\mu}_{i-\tau,j-\sigma}\right)\right| \\
\ge{}& \frac{3}{4}am^{\lambda}n^{\mu} - \frac{1}{4}am^{\lambda}n^{\mu} = \frac{1}{2}am^{\lambda}n^{\mu}
\end{aligned}
\tag{3.406}
$$

for $m \ge M$, $n \ge N$. The above observation shows that Sy is well defined on $N_M \times N_N$ and that $Sy \in \Omega$.

Next, we show that S is continuous on Ω. We assume that y^i, $y \in \Omega$, $i = 1,2,\ldots,y^i \to y$ as $i \to \infty$. Let x^i, x be solutions of (3.397) corresponding to y^i and y, respectively. Then by virtue of Lemma 3.81, we find that $x^i \to x$ as $i \to \infty$ on $N_{\widetilde{M}} \times N_{\widetilde{N}}$. By the Lebesgue dominated convergence theorem we conclude that $Sy^i \to Sy$ as $i \to \infty$, which means that S is continuous on Ω.

It is easy to see that $S\Omega$ is relatively compact.

By the Schauder fixed point theorem, S has a fixed point in Ω, that is, there exists a $y \in \Omega$ such that $Sy = y$. Then we easily see that x is a positive solution of (3.360). The proof is complete. □

We can take the case $u^{\lambda\mu}_{m,n} = m^{\lambda}n^{\mu}$ as an example.

Corollary 3.83. *Let* $u^{\lambda\mu}_{m,n} = m^{\lambda}n^{\mu}$. *Condition* (3.362) *becomes that there exist two positive constants* $a < 1$ *and* $b < 1$ *such that*

$$
1-b \le \liminf_{m,n\to\infty} c_{m,n} \le \limsup_{m,n\to\infty} c_{m,n} \le 1-a. \tag{3.407}
$$

Then the conclusion of Theorem 3.78 holds.

Similarly, we can obtain the following conclusion.

Theorem 3.84. *Let λ, μ be integers with $0 \le \lambda \le h-1$, $0 \le \mu \le r-1$. Further, assume that there exist two positive functions $u^{\lambda\mu}$ and w such that*

$$-\infty < \liminf_{m,n\to\infty} \frac{u^{\lambda\mu}_{m,n} - c_{m,n}u^{\lambda\mu}_{m-k,n-l}}{m^\lambda n^\mu} \le \limsup_{m,n\to\infty} \frac{u^{\lambda\mu}_{m,n} - c_{m,n}u^{\lambda\mu}_{m-k,n-l}}{m^\lambda n^\mu} < 0, \tag{3.408}$$

$$w_{m,n} - c_{m,n}w_{m-k,n-l} = 0, \quad m \ge m_0,\ n \ge n_0. \tag{3.409}$$

Then (3.360) *has a positive solution satisfying*

$$\liminf_{m,n\to\infty} \frac{x_{m,n}}{w_{m,n}} > 0, \tag{3.410}$$

$$\frac{x_{m,n} - c_{m,n}x_{m-k,n-l}}{m^\lambda n^\mu} \longrightarrow \text{const} < 0, \quad m,n \longrightarrow \infty, \tag{3.411}$$

if and only if

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} i^{h-\lambda-1}j^{r-\mu-1}\left| f\left(i,j,aw_{i-\tau,j-\sigma}\right)\right| < \infty, \quad a > 0. \tag{3.412}$$

Moreover, if x is a positive solution of (3.360) *satisfying* (3.410) *and* (3.411), *then*

$$\limsup_{m,n\to\infty} \frac{x_{m,n}}{w_{m,n}} < \infty. \tag{3.413}$$

We need the following lemmas in proving Theorem 3.84.

Lemma 3.85. *Let λ, μ be integers with $0 \le \lambda \le h-1$, $0 \le \mu \le r-1$. The following three statements (i)–(iii) are equivalent.*

(i) *There exists a positive function x satisfying* (3.411).

(ii) *There exists a positive function $u^{\lambda\mu}$ satisfying* (3.408).

(iii) *There exists a positive function $v^{\lambda\mu}$ satisfying*

$$\frac{v^{\lambda\mu}_{m,n} - c_{m,n}v^{\lambda\mu}_{m-k,n-l}}{m^\lambda n^\mu} = -1 \quad \text{for all large } m,n. \tag{3.414}$$

Proof. It is clear that (i) implies (ii), and (iii) implies (i). We will prove that (ii) implies (iii). Suppose that (ii) holds. Then there exists a positive constant $c > 0$ and $M \ge m_0$, $N \ge n_0$ such that

$$\frac{u^{\lambda\mu}_{m,n}}{m^\lambda n^\mu} - c_{m,n}\frac{(m-k)^\lambda(n-l)^\mu}{m^\lambda n^\mu}\cdot\frac{u^{\lambda\mu}_{m-k,n-l}}{(m-k)^\lambda(n-l)^\mu} \le -c, \quad m \ge M,\ n \ge N. \tag{3.415}$$

We can choose a function $\varphi : R_{-1}(M,N) \to R$ such that

$$\frac{u^{\lambda\mu}_{m,n}}{m^\lambda n^\mu} \le -c\varphi_{m,n}, \quad (m,n) \in R_{-1}(M,N),$$
$$\varphi_{M,N} - c_{M,N}\frac{(M-k)^\lambda(N-l)^\mu}{M^\lambda N^\mu}\varphi_{M-k,N-l} = 1. \tag{3.416}$$

Let $\widetilde{v} : \bigcup_{i=-1}^{\infty} R_i(M,N) \to R$ be a solution of the IVP

$$\widetilde{v}_{m,n} - c_{m,n}\frac{(m-k)^\lambda(n-l)^\mu}{m^\lambda n^\mu}\widetilde{v}_{m-k,n-l} = -c, \quad m \ge M,\ n \ge N,$$
$$\widetilde{v}_{m,n} = -c\varphi_{m,n}, \quad (m,n) \in R_{-1}(M,N). \tag{3.417}$$

We see from Lemma 3.80 that

$$\widetilde{v}_{m,n} \ge \frac{u^{\lambda\mu}_{m,n}}{m^\lambda n^\mu} > 0, \quad m \ge M-k,\ n \ge N-l. \tag{3.418}$$

Then $v^{\lambda\mu}_{m,n} = (1/c)m^\lambda n^\mu \widetilde{v}_{m,n}$ is a positive function defined on $\bigcup_{i=-1}^{\infty} R_i(M,N)$ and satisfies

$$\frac{v^{\lambda\mu}_{m,n} - c_{m,n}v^{\lambda\mu}_{m-k,n-l}}{m^\lambda n^\mu} = -1, \quad m \ge M,\ n \ge N. \tag{3.419}$$

Hence (iii) holds. The proof is complete. □

Lemma 3.86. *Let both w^1 and w^2 be positive functions on $N_{M_0-k} \times N_{N_0-l}$ which satisfy* (3.382). *Then there exist positive constants c_* and c^* such that*

$$c_* w^2_{m,n} \le w^1_{m,n} \le c^* w^2_{m,n} \quad \text{for } m \ge M_0-k,\ n \ge N_0-l. \tag{3.420}$$

Lemma 3.86 is a direct corollary of Lemma 3.79.

Lemma 3.87. *Let w and $u^{\lambda\mu}$ be positive functions which satisfy* (3.408) *and* (3.397), $0 \le \lambda \le h-1$, $0 \le \mu \le r-1$. *Then there exist constants $c^* > 0$, $M \ge m_0$, $N \ge n_0$ such that*

$$u^{\lambda\mu}_{m,n} \le c^* w_{m,n}, \quad m \ge M,\ n \ge N. \tag{3.421}$$

Proof. By (3.408), there are $M \ge m_0$, $N \ge n_0$ such that

$$\frac{u^{\lambda\mu}_{m,n}}{m^\lambda n^\mu} - c_{m,n}\frac{(m-k)^\lambda(n-l)^\mu}{m^\lambda n^\mu}\cdot\frac{u^{\lambda\mu}_{m-k,n-l}}{(m-k)^\lambda(n-l)^\mu} < 0, \quad m \ge M,\ n \ge N. \tag{3.422}$$

We note that

$$\frac{w_{m,n}}{m^\lambda n^\mu} - c_{m,n}\frac{(m-k)^\lambda (n-l)^\mu}{m^\lambda n^\mu} \cdot \frac{w_{m-k,n-l}}{(m-k)^\lambda (n-l)^\mu} = 0, \quad m \geq M,\ n \geq N. \tag{3.423}$$

For a sufficiently large number $c^* > 0$, we have

$$\frac{u_{m,n}^{\lambda\mu}}{m^\lambda n^\mu} \leq \frac{c^* w_{m,n}}{m^\lambda n^\mu}, \quad (m,n) \in R_{-1}(M,N). \tag{3.424}$$

From Lemma 3.79, we obtain

$$\frac{u_{m,n}^{\lambda\mu}}{m^\lambda n^\mu} \leq \frac{c^* w_{m,n}}{m^\lambda n^\mu}, \quad m \geq M-k,\ n \geq N-l. \tag{3.425}$$

Hence the proof is complete. □

*P*roof of Theorem 3.84. Assume that $x_{m,n}$ is a positive solution of (3.322) satisfying (3.388)-(3.389). Since $x_{m,n}$ satisfies (3.389), applying Lemma 3.87 to the case $u^{\lambda\mu} = x$, we obtain

$$x_{m,n} \leq c^* w_{m,n} \quad \text{for all large } m,n, \tag{3.426}$$

where c^* is a positive constant. Thus we get (3.408). On the other hand, by (3.388), there exists a positive constant c_* satisfying

$$c_* w_{m,n} \leq x_{m,n} \quad \text{for all large } m,n. \tag{3.427}$$

As in the proof of Theorem 3.78, we can show that (3.397) holds.

Conversely, we suppose that (3.397) holds. Since we assume the existence of a positive function $u^{\lambda\mu}$ satisfying (3.381), by Lemma 3.85, there is a positive function $v^{\lambda\mu}$ satisfying (3.409). Using Lemma 3.87 in the case $u^{\lambda\mu} = v^{\lambda\mu}$, we find that

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} i^{h-\lambda-1} j^{r-\mu-1} \left| f\left(i,j,c_1\left(v_{i-\tau,j-\sigma}^{\lambda\mu} + w_{i-\tau,j-\sigma}\right)\right)\right| < \infty \tag{3.428}$$

for some $c_1 > 0$. By Lemma 3.87, there exist $M_* > m_0$, $N_* > n_0$, $c_2 > c_1$ such that

$$(c_2 - c_1) v_{m,n}^{\lambda\mu} \leq \frac{1}{3} c_1 w_{m,n}, \quad m \geq M_*,\ n \geq N_*. \tag{3.429}$$

Choose $M \geq m_0$, $N \geq n_0$ so large that

$$\widetilde{M} \equiv \min\{M-k, M-\tau\} \geq M_*, \qquad \tilde{N} \equiv \min\{N-l, N-\sigma\} \geq N_*, \tag{3.430}$$

$$\begin{aligned}\sum_{i=M}^{\infty}\sum_{j=N}^{\infty} & i^{h-\lambda-1} j^{r-\mu-1} \left| f\left(i,j,c_1\left[v_{i-\tau,j-\sigma}^{\lambda\mu}+w_{i-\tau,j-\sigma}\right]\right)\right| \\ &< \frac{1}{2}(c_2-c_1)\lambda!(h-\lambda-1)!\mu!(r-\mu-1)!.\end{aligned} \tag{3.431}$$

Define the set Ω by

$$\Omega = \left\{y : \bigcup_{i=-1}^{\infty} R_i(M,N) \longrightarrow R \mid c_1 m^{\lambda} n^{\mu} \leq y_{m,n} \leq c_2 m^{\lambda} n^{\mu},\ m \geq M,\ n \geq N\right\}. \tag{3.432}$$

For $y \in \Omega$, let x be a solution of

$$\begin{aligned} x_{m,n} - c_{m,n} x_{m-k,n-l} &= -y_{m,n}, \quad m \geq M,\ n \geq N, \\ x_{m,n} &= \frac{y_{M,N}}{M^{\lambda}N^{\mu}} v_{m,n}^{\lambda\mu} + \frac{2}{3} c_1 w_{m,n}, \quad (m,n) \in R_{-1}(M,N).\end{aligned} \tag{3.433}$$

Clearly Ω is a nonempty, closed, and convex set of Banach space X and x is uniquely determined by $y \in \Omega$.

Let $\tilde{c} = (c_1 + c_2)/2$ and define the operator F by

$$\begin{aligned} F y_{m,n} = \tilde{c} - (-1)^{h+r-1} & \frac{1}{(h-1)!(r-1)!} \\ \times \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} & (i-m+h-1)^{(h-1)}(j-n+r-1)^{r-1} \\ & \times f\left(i,j,x_{i-\tau,j-\sigma}\right), \quad m \geq M,\ n \geq N,\end{aligned} \tag{3.434}$$

for $\lambda = \mu = 0$,

$$\begin{aligned} F y_{m,n} &= \tilde{c} m^{\lambda} n^{\mu} - (-1)^{h+r-\lambda-\mu-1} \\ &\times \sum_{s=M}^{m-1}\sum_{t=N}^{n-1} \frac{(m-s)^{(\lambda-1)}(n-t)^{(\mu-1)}}{(\lambda-1)!(\mu-1)!} \\ &\times \sum_{i=s}^{\infty}\sum_{j=t}^{\infty} \frac{(i-s+h-\lambda-1)^{(h-\lambda-1)}(j-n+r-\mu-1)^{r-\mu-1}}{(h-\lambda-1)!(r-\mu-1)!} \\ &\times f\left(i,j,x_{i-\tau,j-\sigma}\right), \quad m \geq M,\ n \geq N,\end{aligned} \tag{3.435}$$

for at least one of $\lambda, \mu \neq 0$. We show that Fy is well defined for $y \in \Omega$ and $F\Omega \subset \Omega$. For $y \in \Omega$, we see that

$$\frac{x_{m,n}}{m^\lambda n^\mu} - c_{m,n}\frac{(m-k)^\lambda (n-l)^\mu}{m^\lambda n^\mu} \cdot \frac{x_{m-k,n-l}}{(m-k)^\lambda (n-l)^\mu} = -\frac{y_{m,n}}{m^\lambda n^\mu}, \quad m \geq M,\ n \geq N,$$
$$\frac{x_{m,n}}{m^\lambda n^\mu} = \frac{y_{M,N}}{M^\lambda N^\mu} \cdot \frac{v_{m,n}^{\lambda\mu}}{m^\lambda n^\mu} + \frac{2}{3} \cdot \frac{c_1 w_{m,n}}{m^\lambda n^\mu}, \quad (m,n) \in R_{-1}(M,N). \tag{3.436}$$

By virtue of (3.429) and (3.430), we observe that

$$\frac{c_2 v_{m,n}^{\lambda\mu}}{m^\lambda n^\mu} + \frac{1}{3} \cdot \frac{c_1 w_{m,n}}{m^\lambda n^\mu} \leq \frac{y_{M,N} v_{m,n}^{\lambda\mu}}{M^\lambda N^\mu m^\lambda n^\mu} + \frac{2}{3} \cdot \frac{c_1 w_{m,n}}{m^\lambda n^\mu} \leq \frac{c_1 v_{m,n}^{\lambda\mu}}{m^\lambda n^\mu} + \frac{c_1 w_{m,n}}{m^\lambda n^\mu}. \tag{3.437}$$

Let

$$\widetilde{x}_{m,n} = c_2 v_{m,n}^{\lambda\mu} + \frac{1}{3} c_1 w_{m,n}, \quad m \geq \widetilde{M},\ n \geq \widetilde{N}. \tag{3.438}$$

Then, for $m \geq M$, $n \geq N$, we have

$$\frac{\widetilde{x}_{m,n}}{m^\lambda n^\mu} - c_{m,n}\frac{(m-k)^\lambda (n-l)^\mu}{m^\lambda n^\mu} \cdot \frac{\widetilde{x}_{m-k,n-l}}{(m-k)^\lambda (n-l)^\mu} = -c_2 \leq -\frac{y_{m,n}}{m^\lambda n^\mu}. \tag{3.439}$$

From Lemma 3.79, we have

$$x_{m,n} \geq \widetilde{x}_{m,n} = c_2 v_{m,n}^{\lambda\mu} + \frac{1}{3} c_1 w_{m,n}, \quad m \geq \widetilde{M},\ n \geq \widetilde{N}. \tag{3.440}$$

In a similar way, we obtain

$$x_{m,n} \leq c_1 v_{m,n}^{\lambda\mu} + c_1 w_{m,n}, \quad m \geq \widetilde{M},\ n \geq \widetilde{N}. \tag{3.441}$$

By (3.440), we see that x is positive for $m \geq \widetilde{M}$, $n \geq \widetilde{N}$. Furthermore, by (3.428) and (3.441), Fy is well defined for all $y \in \Omega$. From (3.431) and (3.441), it follows that

$$\begin{aligned} Fy_{m,n} &\leq \widetilde{c} m^\lambda n^\mu + \frac{m^\lambda n^\mu}{\lambda!\mu!(h-\lambda-1)!(r-\mu-1)!} \\ &\quad \times \sum_{i=s}^{\infty}\sum_{j=t}^{\infty} i^{h-\lambda-1} j^{r-\mu-1} \left| f\left(i,j,c_1\left[v_{i-\tau,j-\sigma}^{\lambda\mu} + w_{i-\tau,j-\sigma}\right]\right)\right| \\ &\leq \left(\widetilde{c} + \frac{c_2 - c_1}{2}\right) m^\lambda n^\mu = c_2 m^\lambda n^\mu, \quad m \geq M,\ n \geq N, \\ Fy_{m,n} &\geq \left(\widetilde{c} - \frac{c_2 - c_1}{2}\right) m^\lambda n^\mu, \quad m \geq M,\ n \geq N, \end{aligned} \tag{3.442}$$

which implies that $Fy \in \Omega$.

As in the proof of Theorem 3.78, the continuity of F and the relatively compactness of $F\Omega$ are verified. The Schauder fixed point theorem implies that F has a fixed point in Ω and the x corresponding y is a solution of (3.322). Obviously, we can see that x satisfies (3.388). The proof is complete. □

Next, we will assume that there exists a function $F : N_{m_0} \times N_{n_0} \times R$ such that

$$|f(m,n,u)| \le F(m,n,|u|), \quad (m,n,u) \in N_{m_0} \times N_{n_0} \times R, \tag{3.443}$$

and for all $(m,n) \in N_{m_0} \times N_{n_0}$, $F(m,n,u)$ is nonincreasing in u for $u \in R$.

Theorem 3.88. *Let* $0 \le c_{m,n} \le c_0 < 1$ *and* λ, μ *be integers with* $0 \le \lambda \le h-1$, $0 \le \mu \le r-1$. *If*

$$\sum_{m=M}^{\infty} \sum_{n=N}^{\infty} m^{h-\lambda-1} n^{r-\mu-1} F(m,n,c(m-\tau)^{\lambda}(n-\sigma)^{\mu}) < \infty \quad \text{for some } c > 0, \tag{3.444}$$

then (3.322) *has an eventually positive solution* x *satisfying*

$$0 < \liminf_{m,n\to\infty} \frac{x_{m,n}}{m^{\lambda} n^{\mu}} \le \limsup_{m,n\to\infty} \frac{x_{m,n}}{m^{\lambda} n^{\mu}} < \infty. \tag{3.445}$$

For this result, we prepare the following lemma.

Lemma 3.89. *Let* $0 \le c_{m,n} \le c_0 < 1$ *and* λ, μ *be integers with* $0 \le \lambda \le h-1$, $0 \le \mu \le r-1$. *Let* $u^{\lambda\mu}$ *be a positive function satisfying* (3.323). *Then*

$$0 < \liminf_{m,n\to\infty} \frac{\overset{\lambda\mu}{u}_{m,n}}{m^{\lambda} n^{\mu}} \le \limsup_{m,n\to\infty} \frac{\overset{\lambda\mu}{u}_{m,n}}{m^{\lambda} n^{\mu}} < \infty. \tag{3.446}$$

Proof. Define $\overset{\lambda\mu}{v}_{m,n} = m^{\lambda} n^{\mu}$. We observe that

$$\lim_{m,n\to\infty} \frac{\overset{\lambda\mu}{v}_{m,n} - c_0 \overset{\lambda\mu}{v}_{m-k,n-l}}{m^{\lambda} n^{\mu}} = 1 - c_0. \tag{3.447}$$

Then we can choose sufficiently large M, N, a sufficiently small $c_* > 0$, and a sufficiently large $c^* > 0$ such that

$$\begin{gathered} c_* \frac{\overset{\lambda\mu}{v}_{m,n}}{m^{\lambda} n^{\mu}} \le \frac{\overset{\lambda\mu}{u}_{m,n} - c_{m,n} \overset{\lambda\mu}{u}_{m-k,n-l}}{m^{\lambda} n^{\mu}} \le c^* \frac{\overset{\lambda\mu}{v}_{m,n} - c_0 \overset{\lambda\mu}{v}_{m-k,n-l}}{m^{\lambda} n^{\mu}}, \quad m \ge M,\ n \ge N, \\ c_* \overset{\lambda\mu}{v}_{m,n} \le \overset{\lambda\mu}{u}_{m,n} \le c^* \overset{\lambda\mu}{v}_{m,n}, \quad (m,n) \in R_{-1}(M,N). \end{gathered} \tag{3.448}$$

It follows that

$$c_* v_{m,n}^{\lambda\mu} \le u_{m,n}^{\lambda\mu} - c_{m,n} u_{m-k,n-l}^{\lambda\mu} \le c^* v_{m,n}^{\lambda\mu} - c_0 c^* v_{m-k,n-l}^{\lambda\mu}, \quad m \ge M,\ m \ge N,$$
$$c_* v_{m,n}^{\lambda\mu} \le u_{m,n}^{\lambda\mu} \le c^* v_{m,n}^{\lambda\mu}, \quad (m,n) \in R_{-1}(M,N). \tag{3.449}$$

So we have

$$c_* \le \frac{u_{m,n}^{\lambda\mu}}{v_{m,n}^{\lambda\mu}} \le c^*, \quad m \ge M-k,\ n \ge N-l. \tag{3.450}$$

The proof is complete. □

*P*roof of Theorem 3.88. From Theorem 3.78, we have that under the condition of Theorem 3.88

$$c_* \le \frac{x_{m,n}}{u_{m,n}^{\lambda\mu}} \le c^* \quad \text{for all large } m,n. \tag{3.451}$$

By Lemma 3.89, we obtain

$$c_* \le \frac{u_{m,n}^{\lambda\mu}}{v_{m,n}^{\lambda\mu}} \le c^* \quad \text{for all large } m,n. \tag{3.452}$$

Hence

$$0 < \liminf_{m,n\to\infty} \frac{x_{m,n}}{m^\lambda n^\mu} \le \limsup_{m,n\to\infty} \frac{x_{m,n}}{m^\lambda n^\mu} < \infty. \tag{3.453}$$

The proof is complete. □

3.6. Application in population models

In order to describe the population of the Australian sheep blowfly that agrees well with the experimental data of Nicholson, Gurney et al. proposed the following nonlinear delay differential equation:

$$P'(t) = -\delta P(t) + qP(t-\sigma)e^{-aP(t-\sigma)}, \tag{3.454}$$

where $P(t)$ is the size of the population at time t, q is the maximum per capita daily egg production, $1/a$ is the size at which the blowfly population reproduces at its maximum rate, δ is the pair capita daily adult death rate, and σ is the generation time. Since this equation explains Nicholson's data of blowfly quite accurately, it is now referred to as the Nicholson's blowflies model.

The discrete analog of (3.454) is the delay difference equation

$$P_{n+1} - P_n = -\delta P_n + qP_{n-\sigma}e^{-aP_{n-\sigma}}, \tag{3.455}$$

where $q, a \in (0, \infty)$, $\delta \in (0, 1)$ and $n \in N$. The state variable P_n in (3.455) represents the number of sexually mature blowflies in cycle n as a closed system of the mature flies surviving from previous cycles plus the flies which have survived from the previous σ cycle. Specifically, $qP_{n-\sigma}e^{-aP_{n-\sigma}}$ represents the number of mature flies that were born in the $(n - \sigma)$th cycle and survived to maturity in the nth cycle.

In this section, we will consider the nonlinear delay partial difference equation

$$P_{m+1,n} + P_{m,n+1} - P_{m,n} = -\delta P_{m,n} + qP_{m-\sigma,n-\tau}e^{-aP_{m-\sigma,n-\tau}}, \quad (m, n) \in N_0^2, \tag{3.456}$$

where $q, a \in (0, \infty)$, $\delta \in (0, 1)$, $q > e(1+\delta)$, σ and $\tau \in N_1$. Here $P_{m,n}$ represents the number of the population of blowflies at time m and site n. Let $\Omega = N_{-\sigma} \times N_{-\tau} \setminus N_0 \times N_1$. Given a function $\phi_{m,n}$ defined on Ω, it is easy to construct by induction a double sequence $\{P_{m,n}\}$ which equals $\phi_{m,n}$ on Ω and satisfies (3.456) on $N_0 \times N_1$. Such a double sequence is unique and is said to be a solution of (3.456) with the initial condition

$$P_{m,n} = \phi_{m,n}, \quad (m, n) \in \Omega. \tag{3.457}$$

We say that P^* is an equilibrium of (3.456) if

$$P^* = -\delta P^* + qP^* e^{-aP^*}. \tag{3.458}$$

From (3.458) it is clear that there are two equilibria for (3.458), $P_0 = 0$, which represents extinction, and a positive equilibrium

$$P^* = \frac{1}{a} \ln\left(\frac{q}{1+\delta}\right), \tag{3.459}$$

provided that $q > e(1 + \delta)$.

A solution $\{P_{m,n}\}$ of (3.456) is said to be eventually positive if $P_{m,n} > 0$ for all large m and n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. We consider only such positive solutions of (3.456), which are nontrivial for all large m, n.

A solution $\{P_{m,n}\}$ of (3.456) is said to oscillate about the equilibrium P^* if the terms $P_{m,n} - P^*$ of the sequence $\{P_{m,n} - P^*\}$ are neither all eventually positive nor all eventually negative.

We will show that every positive solution of (3.456) which does not oscillate about the positive equilibrium point P^* converges to P^* as $m, n \to \infty$ and present some sufficient conditions for oscillation of all positive solutions of (3.456) about P^*.

Theorem 3.90. *Suppose that $\delta \in (0,1]$, $\beta \geq \delta$, $q \geq 0$, σ, τ are positive integers, and that $f : R \to R$ is nondecreasing function. Suppose further that $xf(x) > 0$ for all $x \neq 0$. Then every nonoscillatory solution of the nonlinear delay difference equation*

$$x_{m+1,n} + \beta x_{m,n+1} - \delta x_{m,n} + qf(x_{m-\sigma,n-\tau}) = 0 \tag{3.460}$$

tends to zero as $m, n \to \infty$.

Proof. We will consider only the case when $\{x_{m,n}\}$ is eventually positive as the arguments when $\{x_{m,n}\}$ is eventually negative are similar and hence omitted. Suppose that there exist $m_0 > 0$ and $n_0 > 0$ sufficiently large such that $x_{m+1,n} > 0$, $x_{m,n+1} > 0$, $x_{m,n} > 0$, $x_{m-\sigma,n-\tau} > 0$ for $m \geq m_0$ and $n \geq n_0$. Then, from (3.460) we have

$$x_{m+1,n} + \beta x_{m,n+1} \leq \delta x_{m,n}, \quad m \geq m_0,\ n \geq n_0, \tag{3.461}$$

from which it follows that

$$x_{m+1,n} \leq \delta x_{m,n}, \quad x_{m,n+1} \leq \frac{\delta}{\beta} x_{m,n}, \quad m \geq m_0,\ n \geq n_0. \tag{3.462}$$

Now, since $\beta \geq \delta$ and $\delta \leq 1$, then (3.462) implies that

$$x_{m+1,n} \leq x_{m,n}, \quad x_{m,n+1} \leq x_{m,n},\ m \geq m_0,\ n \geq n_0, \tag{3.463}$$

and then $\{x_{m,n}\}$ is nonincreasing sequence in both m and n; thus $x_{m,n} \to b \geq 0$ as $m, n \to \infty$. We assert that $b = 0$. If not, there exist $m_1 \geq m_0$ and $n_1 \geq n_0$ sufficiently large such that $x_{m,n+1} \geq b > 0$, $x_{m+1,n} \geq b > 0$, $x_{m,n} > b$ and $x_{m-\sigma,n-\tau} \geq b > 0$ for $m \geq M = m_1 + \sigma$ and $n \geq N = n_1 + \tau$. Now, since f is a nondecreasing function then $f(x_{m-\sigma,n-\tau}) \geq f(b)$. From (3.460), we have

$$x_{m+1,n} + \beta x_{m,n+1} - \delta x_{m,n} \leq -qf(b). \tag{3.464}$$

Now, it follows from Lemma 2.74 that

$$\sum_{i=M}^{m} \sum_{j=N}^{n} (x_{i+1,j} + \beta x_{i,j+1} - \delta x_{i,j}) \geq x_{m,n+1} - \delta x_{M,N} \geq b - \delta x_{M,N} \tag{3.465}$$

for large M and N. On the other hand, from (3.464) it is clear that for large M and N,

$$\sum_{i=M}^{m} \sum_{j=N}^{n} (x_{i+1,j} + \beta x_{i,j+1} - \delta x_{i,j}) \leq -qf(b)(m-M)(n-N). \tag{3.466}$$

Combining (3.465) and (3.466), we get

$$0 \geq b - \delta x_{M,N} + qf(b)(m-M)(n-N). \tag{3.467}$$

If $b > 0$, then the right-hand side of (3.467) tends to infinity as $m, n \to \infty$ and this leads to a contradiction. Hence $b = 0$. This completes the proof. □

Now, we are ready to state our main results.

Theorem 3.91. *Let $\{P_{m,n}\}$ be a positive solution of* (3.456) *which does not oscillate about P^*. Then $P_{m,n}$ tends to P^* as $m, n \to \infty$.*

Proof. Let $\{P_{m,n}\}$ be an arbitrary positive solution of (3.456) which does not oscillate about P^* and let

$$P_{m,n} = P^* + \frac{1}{a} z_{m,n}. \tag{3.468}$$

Clearly, $z_{m,n}$ does not oscillate, and satisfies the equation

$$z_{m+1,n} + z_{m,n+1} - pz_{m,n} + aP^*(\delta+1)f_1(z_{m-\sigma,n-\tau}) - (\delta+1)f_2(z_{m-\sigma,n-\tau}) = 0 \tag{3.469}$$

with $0 < p = (1-\delta) < 1$,

$$f_1(u) = 1 - e^{-u}, \qquad f_2(u) = ue^{-u}. \tag{3.470}$$

We observe that $f_1(u) \geq f_2(u)$ for all $u \in R$, $uf_1(u) > 0$ and $uf_2(u) > 0$ for all $u \neq 0$. Since $P^* = (1/a)\ln(q/(\delta+1))$, thus, in this case, the condition $aP^* > 1$ is the same as $\ln(q/(\delta+1)) > 1$, that is, $q > e(\delta+1)$, then from (3.469) we have

$$z_{m+1,n} + z_{m,n+1} - pz_{m,n} + (\delta+1)(aP^*-1)f_1(z_{m-\sigma,n-\tau}) \leq 0. \tag{3.471}$$

Note that f_1 is nondecreasing function and that $(\delta+1)(aP^*-1) > 0$. Then, from Corollary 3.27, the equation

$$z_{m+1,n} + z_{m,n+1} - pz_{m,n} + (\delta+1)(aP^*-1)f_1(z_{m-\sigma,n-\tau}) = 0 \tag{3.472}$$

has an eventually positive solution. From (3.470) and since $p < 1$ and $(\delta+1)(aP^* - 1) > 0$, Theorem 3.90 implies that every nonoscillatory solution of (3.472) tends to zero as $m, n \to \infty$. Then from the transformation (3.468), we see that every positive solution of (3.456) tends to P^* as $m, n \to \infty$. The proof is complete. □

Theorem 3.92. *Assume that every solution of the equation*

$$z_{m+1,n} + z_{m,n+1} - pz_{m,n} + (\delta+1)(aP^*-1)z_{m-\sigma,n-\tau} = 0 \tag{3.473}$$

oscillates. Then every positive solution of (3.456) *oscillates about P^*.*

Proof. Assume for the sake of contradiction that (3.456) has a positive solution which does not oscillate about P^*. Without loss of generality, we assume that $P_{m,n} > P^*$ and this implies that $z_{m,n} > 0$. (The case when $P_{m,n} < P^*$ implies that $z_{m,n} < 0$ for which the proof is similar, since $uf_1(u) > 0$ for all $u \neq 0$.) Again define $z_{m,n}$ as in (3.468). Then, from the proof of Theorem 3.91 it is clear that $z_{m,n} > 0$ and satisfies (3.472). From (3.470) we observe that f_1 is nondecreasing function, $(\delta + 1)(aP^* - 1) > 0$,

$$uf_1(u) > 0 \quad \text{for } u \neq 0, \quad \lim_{u\to 0} \frac{f_1(u)}{u} = 1. \tag{3.474}$$

Also we claim that

$$f_1(u) \leq u \quad \text{for } u > 0. \tag{3.475}$$

The proof of (3.475) follows from the observation that $f_1(0) = 0$ and that

$$\frac{d}{du}\left(f_1(u) - u\right) = -\left(1 - \frac{1}{e^u}\right) < 0 \quad \text{for } u > 0. \tag{3.476}$$

Then (3.475) holds. Then, from (3.474) and (3.475), Theorem 3.8 implies that there exists an eventually positive solution of (3.473). This contradiction shows that every positive solution of (3.456) oscillates about P^*.

Theorem 3.92 shows that the oscillation of every positive solution of (3.456) about P^* is equivalent to the oscillation of the delay difference equation (3.473). Thus, we can use the result in Section 2.2 to obtain an oscillation criterion. We state such a result in the following theorem. □

Theorem 3.93. *Every positive solution of* (3.456) *oscillates about* P^* *if and only if*

$$(\delta + 1)(aP^* - 1)\frac{(\sigma + \tau + 1)^{\sigma+\tau+1}}{\sigma^\sigma \tau^\tau (1 - \delta)^{\sigma+\tau+1}} > 1. \tag{3.477}$$

It remains an open problem to prove that every oscillatory solution of (3.456) *tends to* P^* *as* $m, n \to \infty$ *to complete the proof of global attractivity.*

Next, we will consider the discrete partial delay survival red blood cells model

$$P_{m+1,n} + P_{m,n+1} - P_{m,n} = -\delta P_{m,n} + qe^{-aP_{m-\sigma,n-\tau}}, \tag{3.478}$$

where $P_{m,n}$ represents the number of the red blood cells at time m and site n, $\delta \in (0, 1)$, a and q are positive constants and σ and τ are positive integers. We will show that (3.478) has a unique positive steady state P^* and that every positive solution of (3.478) which does not oscillate about P^* converges to P^* as $m, n \to \infty$, and present necessary and sufficient conditions for oscillation of all positive solutions of (3.478) about P^*.

Let $\Omega = N_{-\sigma} \times N_{-\tau} \setminus N_0 \times N_1$. Given a function $\phi_{m,n}$ defined on Ω, it is easy to construct by induction a double sequence $\{P_{m,n}\}$ which equals $\phi_{m,n}$ on Ω and satisfies (3.478) on $N_0 \times N_1$. We say that P^* is an equilibrium of (3.478) if

$$P^* = -\delta P^* + qe^{-aP^*}. \tag{3.479}$$

Now, we prove that (3.478) has a unique equilibrium P^*. Observe that the equilibrium points of (3.478) are the solutions of the equation

$$qe^{-aP^*} - (1+\delta)P^* = 0. \tag{3.480}$$

Set

$$f(x) = qe^{-ax} - (1+\delta)x, \tag{3.481}$$

then $f(0)=q>0$ and $f(\infty)=-\infty$, so that there exists $x^* > 0$ such that $f(x^*) = 0$. Now since $f'(x) = -aqe^{-ax} - \delta < 0$ for all $x > 0$, then $f'(x^*) < 0$, from which it follows that $f(x) = 0$ has exactly one solution x^*, and then (3.478) has a unique equilibrium point P^*.

Theorem 3.94. *Let* $\{P_{m,n}\}$ *be a positive solution of* (3.478) *which does not oscillate about* P^*. *Then* $P_{m,n}$ *tends to* P^* *as* $m, n \to \infty$.

Proof. Let $\{P_{m,n}\}$ be an arbitrary positive solution of (3.478) which does not oscillate about P^* and let

$$P_{m,n} = P^* + \frac{1}{a} z_{m,n}. \tag{3.482}$$

Clearly, $z_{m,n}$ does not oscillate, and satisfies the equation

$$z_{m+1,n} + z_{m,n+1} - pz_{m,n} + qae^{-aP^*} f(z_{m-\sigma,n-\tau}) = 0, \tag{3.483}$$

where

$$0 < p = 1 - \delta < 1, \qquad f(u) = 1 - e^{-u}. \tag{3.484}$$

Note that, f is a nondecreasing function,

$$uf(u) > 0 \quad \text{for } u \neq 0, \quad \lim_{u\to 0} \frac{f(u)}{u} = 1. \tag{3.485}$$

From (3.485) and since $p < 1$ and $qae^{-aP^*} > 0$, Theorem 3.90 implies that every nonoscillatory solution of (3.483) tends to zero as $m, n \to \infty$. Then from the transformation (3.482), we see that every positive solution of (3.478) which does not oscillate about P^* tends to P^* as $m, n \to \infty$. The proof is complete. □

Theorem 3.95. *Then every positive solution of* (3.478) *oscillates about* P^* *if and only if*

$$qae^{-aP^*}\frac{(\sigma+\tau+1)^{\sigma+\tau+1}}{\sigma^\sigma\tau^\tau(1-\delta)^{\sigma+\tau+1}} > 1. \tag{3.486}$$

Proof. Assume for the sake of contradiction that (3.478) has a positive solution which does not oscillate about P^*. Without loss of generality, we assume that $P_{m,n} > P^*$ and this implies that $z_{m,n} > 0$. The case when $P_{m,n} < P^*$ implies that $z_{m,n} < 0$ for which the proof is similar. In fact, we see that if $\{z_{m,n}\}$ is a negative solution of (3.483) then $\{U_{m,n}\} = \{-z_{m,n}\}$ is a positive solution of (3.483). From the transformation (3.482) it is clear that $P_{m,n}$ oscillates about P^* if and only if $z_{m,n}$ oscillates about zero. The transformation (3.482) transforms (3.478) to (3.483) and (3.485) holds. Also we claim that

$$f(u) \le u \quad \text{for } u > 0. \tag{3.487}$$

The proof of (3.487) follows from the observation that $f(0) = 0$ and that

$$\frac{d}{du}(f(u)-u) = -\left(1-\frac{1}{e^u}\right) < 0 \quad \text{for } u > 0. \tag{3.488}$$

Then (3.487) holds. The linearized equation associated with (3.483) is

$$z_{m+1,n} + z_{m,n+1} - pz_{m,n} + qae^{-aP^*}z_{m-\sigma,n-\tau} = 0. \tag{3.489}$$

Then by Theorem 2.3, every solution of (3.489) oscillates if and only if (3.486) holds. The proof is now elementary consequence of the linearized oscillation Theorem 3.9 according to which the following statements are true. If (3.485) and (3.487) hold, then every solution of (3.483) oscillates if and only if every solution of (3.489) oscillates. Thus, in conclusion, every positive solution of (3.478) oscillates about P^*. □

3.7. Oscillations of initial boundary value problems

3.7.1. Parabolic equations

Consider delay partial difference equations of the form

$$\Delta_2 u_{i,j} = a_j\Delta_1^2 u_{i-1,j} - q_{i,j}f(u_{i,j-\sigma}), \quad 1 \le i \le n,\ j \ge 0, \tag{3.490}$$

where the delay σ is a nonnegative integer, $a_j > 0$ for $j \ge 0$ and f is a real function on R. The real function $u_{i,j}$ is dependent on integral variables i, j which satisfy $0 \le i \le n+1$ and $j \ge -\sigma$. In (3.490), we use the following notations:

$$\begin{aligned}\Delta_2 u_{i,j} &= u_{i,j+1} - u_{i,j}, \qquad \Delta_1 u_{i,j} = u_{i+1,j} - u_{i,j},\\ \Delta_1^2 u_{i-1,j} &= \Delta_1(\Delta_1 u_{i-1,j}) = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}.\end{aligned} \tag{3.491}$$

We assume that $u_{i,j}$ is subject to the conditions

$$u_{0,j} + \alpha_j u_{1,j} = 0, \quad j \geq 0, \tag{3.492}$$

$$u_{n+1,j} + \beta_j u_{n,j} = 0, \quad j \geq 0, \tag{3.493}$$

$$u_{i,j} = \rho_{i,j}, \quad -\sigma \leq j \leq 0,\ 0 \leq i \leq n+1, \tag{3.494}$$

where $\alpha_j + 1 \geq 0$ and $\beta_j + 1 \geq 0$ for $j \geq 0$.

Given an arbitrary function $\rho_{i,j}$ which is defined for $-\sigma \leq j \leq 0$ and $0 \leq i \leq n+1$ and arbitrary functions α_j and β_j for $j \geq 0$, we can show that a solution to (3.490)–(3.494) exists and is unique. In fact, from (3.490), we have

$$\begin{gathered} u_{i,1} = a_0\rho_{i+1,0} + (1-2a_0)\rho_{i,0} + a_0\rho_{i-1,0} - q_{i,0} f(\rho_{i,-\sigma}), \quad 1 \leq i \leq n, \\ u_{0,1} = -\alpha_1 u_{1,1}, \qquad u_{n+1,1} = -\beta_1 u_{n,1}. \end{gathered} \tag{3.495}$$

Inductively, we see that $\{u_{i,j+1}\}_{i=1}^{n+1}$ is determined uniquely by $\{u_{i,k}\}_{i=0}^{n+1}$, $k \leq j$.

Let $v_{i,j}$ be a real function defined for $0 \leq i \leq n+1$ and $j \geq -\sigma$. Suppose there is some nonnegative integer T such that $v_{i,j} > 0$ for $1 \leq i \leq n$ and $j \geq T$, then $v_{i,j}$ is said to be eventually positive. An eventually negative $v_{i,j}$ is similarly defined. The function $v_{i,j}$ is said to be oscillatory for $1 \leq i \leq n$ and $j \geq 0$, if it is neither eventually positive nor eventually negative.

We now assume that $q_{i,j} \geq 0$ for $1 \leq i \leq n$ and $j \geq 0$. Let

$$Q_j = \min\{q_{i,j} \mid 1 \leq i \leq n\}. \tag{3.496}$$

By the average technique we will prove the following result.

Theorem 3.96. *Let σ be a positive integer and suppose that $Q_j \geq 0$ for $j \geq 0$. Let $f(x)$ be a real function defined on R such that $xf(x) > 0$ for $x \neq 0$, $f(x)$ is nondecreasing on R, $f(x)$ and $-f(-x)$ are convex on* $(0,+\infty)$ *such that*

$$\lim_{x\to 0} \frac{x}{f(x)} = M > 0. \tag{3.497}$$

If

$$\liminf_{n\to\infty} \frac{1}{\sigma} \sum_{j=n-\sigma}^{n-1} Q_j > \frac{M\sigma^\sigma}{(1+\sigma)^{1+\sigma}} \tag{3.498}$$

then every solution of (3.490)–(3.494) *oscillates.*

Proof. Suppose to the contrary, let $\{u_{i,j}\}$ be an eventually positive solution of (3.490) such that $u_{i,j} > 0$ for $1 \leq i \leq n$ and $j \geq T$. From (3.490), we have

$$\frac{1}{n}\sum_{i=1}^{n} \Delta_2 u_{i,j} = \frac{a_j}{n}\sum_{i=1}^{n} \Delta_1^2 u_{i-1,j} - \sum_{i=1}^{n} \frac{q_{i,j}}{n} f(u_{i,j-\sigma}). \tag{3.499}$$

Since f is convex, by Jensen's inequality (1.29), we have

$$\sum_{i=1}^{n} \frac{q_{i,j}}{n} f(u_{i,j-\sigma}) \geq Q_j f\left(\frac{1}{n}\sum_{i=1}^{n} u_{i,j-\sigma}\right). \tag{3.500}$$

By conditions (3.492) and (3.493),

$$a_j \sum_{i=1}^{n} \Delta_1^2 u_{i-1,j} = a_j\left[-(\beta_j+1)u_{n,j} - (\alpha_j+1)u_{1,j}\right] \leq 0 \quad \text{for } j \geq T. \tag{3.501}$$

Let $\omega_j = (1/n)\sum_{i=1}^{n} u_{i,j}$. From (3.499)–(3.501), we have

$$\Delta\omega_j + Q_j f(\omega_{j-\sigma}) \leq 0, \quad j \geq T, \tag{3.502}$$

that is, (3.502) has a positive solution ω_j, $j \geq T$.

In order to complete the proof of Theorem 3.96, we need the following lemmas.

Lemma 3.97. *Assume that the assumptions of Theorem 3.96 hold. If the difference inequality* (3.502) *has an eventually positive solution, then* (3.498) *does not hold.*

Proof. Let $\{\omega_i\}$ be an eventually positive solution of (3.502) such that $\omega_i > 0$ for $i \geq T$. Then $\Delta\omega_i < 0$ for $i \geq T$. Hence $\lim_{i\to\infty}\omega_i$ exists and

$$\omega_i + \sum_{j=T+\sigma}^{i-1} Q_j f(\omega_{j-\sigma}) \leq \omega_T. \tag{3.503}$$

Thus

$$\sum_{j=N}^{\infty} Q_j f(\omega_{j-\sigma}) < \infty. \tag{3.504}$$

If (3.498) holds, then $\sum_{j=N}^{\infty} Q_j = \infty$, which together with (3.504) imply that $\omega_j \to 0$ as $j \to \infty$.

From (3.502),

$$\Delta\omega_n + Q_n f(\omega_n) \leq \Delta\omega_n + Q_n f(\omega_{n-\sigma}) \leq 0, \tag{3.505}$$

so that

$$\begin{gathered} Q_n \leq \frac{\omega_n}{f(\omega_n)}\left(1 - \frac{\omega_{n+1}}{\omega_n}\right), \\ \frac{1}{\sigma}\sum_{i=n-\sigma}^{n-1} Q_i \leq \frac{1}{\sigma}\sum_{i=n-\sigma}^{n-1} \frac{\omega_i}{f(\omega_i)}\left(1 - \frac{\omega_{i+1}}{\omega_i}\right). \end{gathered} \tag{3.506}$$

Set $\chi = M\sigma^\sigma/(\sigma+1)^{\sigma+1}$. If (3.498) holds, then there exists a constant τ such that for sufficiently large n,

$$\chi < \tau \le \frac{1}{\sigma}\sum_{i=n-\sigma}^{n-1} Q_i. \tag{3.507}$$

Since $\omega_j \to 0$ as $j \to \infty$, there exists $\varepsilon > 0$ such that $\varepsilon < (\tau/\chi - 1)M$ and

$$\frac{\omega_n}{f(\omega_n)} \le M + \varepsilon \quad \text{for all large } n. \tag{3.508}$$

From (3.506) and (3.508), by the inequality between the arithmetic and geometric means, we have

$$\tau \le (M+\varepsilon)\frac{1}{\sigma}\sum_{i=n-\sigma}^{n-1}\left(1 - \frac{\omega_{i+1}}{\omega_i}\right) \le (M+\varepsilon)\left[1 - \left(\frac{\omega_n}{\omega_{n-\sigma}}\right)^{1/\sigma}\right]. \tag{3.509}$$

By means of the inequality

$$1 - \lambda \le \left(\frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}\right)^{1/\sigma}\lambda^{-1/\sigma}, \quad 0 < \lambda \le 1, \tag{3.510}$$

we obtain

$$\frac{\omega_n}{\omega_{n-\sigma}} \le \left(1 - \frac{\tau}{M+\varepsilon}\right)^\sigma \le \frac{\chi(M+\varepsilon)}{\tau M} < 1, \tag{3.511}$$

for all large n.

Substituting the above inequality into (3.502), we have

$$\Delta\omega_n + Q_n f\left(\frac{\tau M}{\chi(M+\varepsilon)}\omega_n\right) \le \Delta\omega_n + Q_n f(\omega_{n-\sigma}) \le 0. \tag{3.512}$$

A similar procedure then leads to

$$\frac{\omega_{n-\sigma}}{\omega_n} \ge \left(\frac{\tau M}{\chi(M+\varepsilon)}\right)^2 \quad \text{for } n \ge n_1. \tag{3.513}$$

Inductively, we see that for every positive integer K, there is n_K such that

$$\frac{\omega_{n-\sigma}}{\omega_n} \ge \left(\frac{\tau M}{\chi(M+\varepsilon)}\right)^K \quad \text{for } n \ge n_K. \tag{3.514}$$

On the other hand, (3.498) implies that there is a positive constant c such that for all large n,

$$\sum_{i=n-\sigma}^{n} Q_i > c, \qquad \sum_{i=n-\sigma}^{n'} Q_j \geq \frac{c}{2}, \qquad \sum_{i=n'}^{n} Q_i \geq \frac{c}{2}, \tag{3.515}$$

where $n-\sigma \leq n' \leq n$.

Since $\Delta\omega_i < 0$, from (3.502) we obtain

$$\begin{aligned} \omega_{n'+1} - \omega_{n-\sigma} &= -\sum_{i=n-\sigma}^{n'} Q_i f(\omega_{i-\sigma}) \leq -f(\omega_{n'-\sigma}) \sum_{i=n-\sigma}^{n'} Q_i \leq -\frac{c}{2} f(\omega_{n'-\sigma}), \\ \omega_{n+1} - \omega_{n'} &= -\sum_{i=n'}^{n} Q_i f(\omega_{i-\sigma}) \leq -f(\omega_{n-\sigma}) \sum_{i=n'}^{n} Q_i \leq -\frac{c}{2} f(\omega_{n-\sigma}). \end{aligned} \tag{3.516}$$

Combining (3.516), we obtain

$$0 \geq \omega_{n+1} - \omega_{n'} + \frac{c}{2} f(\omega_{n-\sigma}) \frac{\omega_{n-\sigma}}{\omega_{n-\sigma}} \geq \omega_{n+1} - \omega_{n'} + \frac{c}{2} \frac{f(\omega_{n-\sigma})}{\omega_{n-\sigma}} \left(\omega_{n'+1} + \frac{c}{2} f(\omega_{n'-\sigma})\right), \tag{3.517}$$

so that

$$\begin{gathered} -\omega_{n'} + \left(\frac{c}{2}\right)^2 \frac{f(\omega_{n-\sigma})}{\omega_{n-\sigma}} f(\omega_{n'-\sigma}) \leq 0, \\ \frac{\omega_{n'-\sigma}}{\omega_{n'}} \leq \left(\frac{2}{c}\right)^2 \frac{\omega_{n-\sigma}}{f(\omega_{n-\sigma})} \frac{\omega_{n'-\sigma}}{f(\omega_{n'-\sigma})} \longrightarrow \left(\frac{2}{c}\right)^2 M^2, \end{gathered} \tag{3.518}$$

which contradicts (3.514). In fact, there exists a positive integer K such that

$$\left(\frac{\tau M}{\chi(M+\varepsilon)}\right)^K > \left(\frac{2}{c}\right)^2 M^2. \tag{3.519}$$

The proof of Lemma 3.97 is complete. □

We consider (3.502) together with the equation

$$\Delta\omega_j + Q_j f(\omega_{j-\sigma}) = 0, \quad j \geq 0. \tag{3.520}$$

Lemma 3.98. *Let σ be a positive integer. Suppose that $Q_j \geq 0$ for $j \geq 0$ and*

$$\sum_{j=0}^{\sigma-1} Q_{n+j} > 0 \quad \text{for all large } n. \tag{3.521}$$

Suppose further that $f(x)$ is a positive and nondecreasing function on $(0,\infty)$. If (3.502) has an eventually positive solution, so does (3.520).

Proof. Assume that $\{\omega_j\}$ is an eventually positive solution of (3.502). There exists $T > 0$ such that $\omega_j > 0$ for $j \geq T - 2\sigma$, then $\Delta\omega_j < 0$ for $j \geq T - \sigma$ and

$$\omega_i + \sum_{j=T}^{i-1} Q_j f(\omega_{j-\sigma}) \leq \omega_T. \tag{3.522}$$

Since $\lim_{j\to\infty} \omega_j = \omega_\infty$ exists, so

$$\omega_n \geq \omega_\infty + \sum_{j=n}^{\infty} Q_j f(\omega_{j-\sigma}), \quad n \geq T. \tag{3.523}$$

Let X denote the partially ordered Banach space of all bounded real sequences $\{x_n\}_{n=T}^{\infty}$ with the usual supreme norm and the componentwise defined partial ordering $\leq$. Let Ω be a subset of X defined by

$$\Omega = \{\{x_n\} \in X \mid \omega_\infty \leq x_n \leq \omega_n,\ n \geq T\}. \tag{3.524}$$

For every $x \in \Omega$, define

$$\overline{x}_n = \begin{cases} x_n, & n \geq T, \\ x_T + \omega_n - \omega_T, & T - \sigma \leq n \leq T. \end{cases} \tag{3.525}$$

Note that $0 < \overline{x}_n \leq \omega_n$ for $T - \sigma \leq n < T$.

Define a mapping S on Ω by

$$(Sx)_n = \omega_\infty + \sum_{j=n}^{\infty} Q_j f(\overline{x}_{j-\sigma}), \quad n \geq T. \tag{3.526}$$

By (3.523), $S\Omega \subset \Omega$ and S is monotone. By Knaster-Tarski's fixed point theorem (Theorem 1.9), S has a fixed point $z \in \Omega$. Clearly, z satisfies (3.520) for $j \geq T$, and so the proof will be complete if we can show that $z_n > 0$ for $n \geq T$. In fact, as noted before, $z_n > 0$ for $T - \sigma \leq n < T$. Assume by induction that $z_n > 0$ for $T - \sigma \leq n < K$, where $T \leq K$, then

$$\begin{aligned} z_K &= \omega_\infty + \sum_{j=K}^{\infty} Q_j f(\overline{z}_{j-\sigma}) \\ &\geq \omega_\infty + \sum_{j=K}^{K+\sigma-1} Q_j f(\overline{z}_{j-\sigma}) \\ &\geq \omega_\infty + \min_{K\leq j\leq K+\sigma-1} f(\overline{z}_{j-\sigma}) \sum_{j=K}^{K+\sigma-1} Q_j > 0. \end{aligned} \tag{3.527}$$

The proof of Lemma 3.98 is complete. □

We return to the proof of Theorem 3.96. Suppose that $\{u_{i,j}\}$ is an eventually positive solution of (3.490). By Lemma 3.97, (3.498) does not hold, a contradiction.

We note that $\{u_{i,j}\}$ is a solution of (3.490) if and only if $\{-u_{i,j}\}$ is a solution of

$$\Delta_2 v_{i,j} = a_j \Delta_1^2 v_{i-1,j} - q_{i,j} F(v_{i,j-\sigma}), \quad 1 \le i \le n,\ j \ge 0, \tag{3.528}$$

where $F(t) = -f(-t)$ for all t. Also, $\{\omega_j\}$ is a solution of (3.502) if and only if $\{-\omega_j\}$ is a solution of

$$\Delta z_j + Q_j F(z_{j-\sigma}) \ge 0, \quad j \ge 0. \tag{3.529}$$

Hence if (3.490) has an eventually negative solution, we can derive a contradiction also. The proof of Theorem 3.96 is complete. □

From the proof of Theorem 3.96, we can obtain a more general result.

Corollary 3.99. *Under the assumptions of Theorem 3.96, if every solution of* (3.520) *is oscillatory, then every solution of* (3.490)–(3.494) *is oscillatory.*

Remark 3.100. There are some oscillation criteria for (3.520) in the literature. Therefore we can obtain different oscillation criteria with (3.497) and (3.498) for the oscillation of (3.490).

3.7.2. Nonhomogeneous parabolic equations

We consider the nonhomogeneous partial difference equations of the form

$$\Delta_2 u_{i,j} = a_j \Delta_1^2 u_{i-1,j} - p_j u_{i,j-\sigma} + f_{i,j}, \quad 1 \le i \le n,\ j \ge 0 \tag{3.530}$$

with the conditions

$$\begin{aligned} u_{0,j} &= g_j, \quad j \ge 1, \\ u_{n+1,j} &= h_j, \quad j \ge 1, \\ u_{i,j} &= \varphi_{i,j}, \quad -\sigma \le j \le 0,\ 0 \le i \le n+1. \end{aligned} \tag{3.531}$$

It is easy to prove that (3.530)-(3.531) have a unique solution.

We will be concerned with conditions which imply that every solution of (3.530)-(3.531) is oscillatory. The definition of the oscillation is similar to that in Section 3.7.1.

Theorem 3.101. *Suppose the following conditions hold:*

(i) $a_j \ge 0$ *and* $p_j \ge 0$ *for* $j \ge 0$;

(ii) $\psi_j = a_j(h_j + g_j) + \sum_{i=1}^{n} f_{i,j}$, $j \ge 0$;

(iii) *the difference inequality*

$$\Delta v_j + p_j v_{j-\sigma} \leq (\geq) \psi_j \tag{3.532}$$

has no eventually positive (negative) solutions. Then every solution of (3.530)-(3.531) *is oscillatory.*

Proof. Suppose to the contrary, let $\{u_{i,j}\}$ be an eventually positive solution of (3.530)-(3.531). From (3.530), we have

$$\sum_{i=1}^{n} \Delta_2 u_{i,j} = a_j \sum_{i=1}^{n} \Delta_1^2 u_{i-1,j} - p_j \sum_{i=1}^{n} u_{i,j-\sigma} + \sum_{i=1}^{n} f_{i,j}. \tag{3.533}$$

Since

$$\sum_{i=1}^{n} \Delta_1^2 u_{i-1,j} = \Delta_1 u_{n,j} - \Delta_1 u_{0,j} = (h_j - u_{n,j}) - (u_{1,j} - g_j) \leq h_j + g_j, \tag{3.534}$$

then when $a_j \geq 0$ for all large j, we have

$$\Delta v_j + p_j v_{j-\sigma} \leq a_j (h_j + g_j) + \sum_{i=1}^{n} f_{i,j} = \psi_j \tag{3.535}$$

for all large j, where $v_j = \sum_{i=1}^{n} u_{i,j}$.

Similarly, if (3.530)-(3.531) have an eventually negative solution, then

$$\Delta v_j + p_j v_{j-\sigma} \geq \psi_j \tag{3.536}$$

has an eventually negative solution, which contradicts condition (iii). The proof is complete. □

Now we will show some sufficient conditions for condition (iii).

Theorem 3.102. *Assume that one of the following conditions holds.*

(1) *There exists a sequence* $\{\varphi_j\}$ *such that* $\Delta\varphi_j = \psi_j$ *for* $j \geq T$,

$$\liminf_{j\to\infty} \varphi_j = -\infty, \quad \limsup_{j\to\infty} \varphi_j = +\infty. \tag{3.537}$$

(2) *Assume that*

$$\begin{gathered} \liminf_{j\to\infty} \varphi_j = m > -\infty, \quad \limsup_{j\to\infty} \varphi_j = M < \infty, \\ \sum_{j=T}^{\infty} p_j (\varphi_{j-\sigma} - m)_+ = \infty, \quad \sum_{j=T}^{\infty} p_j (M - \varphi_{j-\sigma})_+ = \infty. \end{gathered} \tag{3.538}$$

Then (iii) holds.

Proof. Suppose to the contrary, let $\{v_j\}$ be an eventually positive solution of (3.535). Then $\Delta(v_j - \varphi_j) < 0$ eventually.

For Case (1), φ_j always changes sign for sufficiently large j. Therefore there exists a sequence $j_K \to \infty$ as $K \to \infty$ such that $v(j_K) > \varphi(j_K)$, and hence $v_j - \varphi_j > 0$ eventually, which implies that $\lim_{j\to\infty}(v_j - \varphi_j) = l \geq 0$ exists. This contradicts the fact $\liminf\ \varphi_j = -\infty$.

Similarly, we can derive a contradiction when (3.536) has an eventually negative solution.

For Case (2), let $\{v_j\}$ be an eventually positive solution of (3.535). As the above $\Delta(v_j - \varphi_j) < 0$, $v_j - \varphi_j > 0$ eventually. Summing the inequality

$$\Delta(v_j - \varphi_j) + p_j v_{j-\sigma} \leq 0, \tag{3.539}$$

we obtain

$$\sum_{j=T}^{\infty} p_j v_{j-\sigma} < \infty. \tag{3.540}$$

By the condition in Case (2), we have $\sum_{j=T}^{\infty} p_j = \infty$. Hence $\liminf_{j\to\infty} v_{j-\sigma} = 0$. Set

$$\lim_{j\to\infty} (v_j - \varphi_j) = l \geq 0. \tag{3.541}$$

We will show that $l = -m$. In fact, for any $\varepsilon > 0$ there exists T such that $l < v_j - \varphi_j < l + \varepsilon$ and hence $-\varphi_j < l + \varepsilon$, $j \geq T$. Then

$$-m = -\liminf_{j\to\infty} \varphi_j \leq l + \varepsilon. \tag{3.542}$$

On the other hand, there exists a sequence $\{j_K\}$ such that $\lim_{K\to\infty} j_K = \infty$ and $\lim_{K\to\infty} v_{j_K} = 0$. From (3.541), we have $-\varphi(j_K) > l - v(j_K)$ and hence $-\liminf_{K\to\infty} \varphi(j_K) \geq l$ and so $-m \geq l$. We have proved that $l = -m$. From (3.541), $v_j - \varphi_j > -m$ and so $v_j > (\varphi_j - m)_+$, $j \geq T$. Then

$$v_{j-\sigma} > (\varphi_{j-\sigma} - m)_+, \quad j \geq T + \sigma. \tag{3.543}$$

Substituting this into (3.540), we obtain

$$\sum_{j=T+\sigma}^{\infty} p_j (\varphi_{j-\sigma} - m)_+ < \infty, \tag{3.544}$$

which contradicts the assumption. Similarly, we can prove that (3.536) has no eventually negative solution. The proof is complete. □

Corollary 3.103. *Assume that (i) and (ii) of Theorem 3.101 hold. Further assume that Condition (1) or Condition (2) of Theorem 3.102 holds. Then every solution of* (3.530)-(3.531) *is oscillatory.*

3.8. Multidimensional initial boundary value problems

3.8.1. Discrete Gaussian formula

Consider a sequence $\{u_{m,n}\} = \{u_{m_1,m_2,\dots,m_\ell,n}\}$ which is defined on $\Omega \times N_{n_0}$, where $\Omega = \{p_1^{(1)}, p_2^{(1)}, \dots, p_{M_1}^{(1)}\} \times \cdots \times \{p_1^{(\ell)}, p_2^{(\ell)}, \dots, p_{M_\ell}^{(\ell)}\}$ and every $p_i^{(j)} \in Z$.

Now we give some definitions for deriving the discrete Gaussian formula.

Definition 3.104. m is said to be an interior point of Ω, if $m+1 \triangleq \{m_1+1, m_2, \dots, m_\ell\} \cup \cdots \cup \{m_1, m_2, \dots, m_{\ell-1}, m_\ell + 1\}$ and $m - 1 \triangleq \{m_1 - 1, m_2, \dots, m_\ell\} \cup \cdots \cup \{m_1, m_2, \dots, m_{\ell-1}, m_\ell - 1\}$ are all in Ω; Ω^0, which is composed of all interior points, is said to be an interior of Ω.

Definition 3.105. m is said to be a convex boundary point of Ω, if $m \in \Omega$ and at least ℓ points of $m \pm 1$ are in Ω; m is said to be a concave boundary point, if $m, m \pm 1 \in \Omega$ but just one of the points $\{m_1 \pm 1, m_2 \pm 1, \dots, m_\ell \pm 1\}$ is not in Ω, where $\{m_1 \pm 1, m_2 \pm 1, \dots, m_\ell \pm 1\} \triangleq \{m_1 + 1, m_2 + 1, \dots, m_\ell + 1\} \cup \{m_1 - 1, m_2 + 1, \dots, m_\ell + 1\} \cup \cdots \cup \{m_1 - 1, m_2 - 1, \dots, m_\ell - 1\} \in \partial\Omega$, which is composed of all (convex and concave) boundary points, is said to be a boundary of Ω.

Remark 3.106. If Ω is a rectangular solid net (its definition can be seen from any book on the computation of partial differential equations), then $\partial\Omega$ is only composed of all convex boundary points.

Definition 3.107. Ω is said to be convex, if $\partial\Omega$ is only composed of all convex points.

It is easy to see that if Ω is a rectangular solid net, then Ω is convex.

Definition 3.108. m is said to be an exterior point, if it is neither an interior point nor a boundary point.

Definition 3.109. m is said to be an allowable point, if at least two points of $m \pm 1$ are in Ω.

Definition 3.110. Ω is said to be a connected net, if Ω is only composed of all allowable points.

Remark 3.111. If Ω is a rectangular solid net, then it is a convex connected solid net.

We only consider in this section that Ω is a convex connected solid net.

Definition 3.112. If $m \in \partial\Omega$ is a convex boundary point of Ω, we define that the normal difference at $(m, n) \in \partial\Omega \times N_{n_0}$ is

$$\Delta_N u_{m-1,n} \triangleq \sum_{\text{all } m\pm 1 \notin \Omega} (\Delta_1 u_{m,n} - \Delta_1 u_{m-1,n}) = \sum_{\text{all } m\pm 1 \notin \Omega} \Delta_1^2 u_{m-1,n}, \tag{3.545}$$

where Δ_1 and Δ_1^2 are, respectively, partial difference operators of order one and of order two.

We write ∇^2 a discrete Laplacian operator, which is defined by

$$\nabla^2 u_{m-1,n+1} \triangleq \sum_{i=1}^{\ell} \Delta_i^2 u_{m_1,\dots,m_{i-1},m_i-1,m_{i+1},\dots,m_\ell,n+1}, \tag{3.546}$$

where Δ_i^2 is a partial difference operator of order two.

Now we give the discrete Gaussian formula as follows.

Theorem 3.113 (discrete Gaussian formula). *Let Ω be a convex connected solid net. Then*

$$\sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} = \sum_{m\in\partial\Omega} \Delta_N u_{m-1,n+1}. \tag{3.547}$$

Proof. Because a convex connected solid net can be divided into several rectangular solid nets, therefore we can only consider the latter case. Without loss of generality, we let $\Omega \triangleq \{1, 2, \dots, M_1\} \times \cdots \times \{1, 2, \dots, M_\ell\}$. In the following we give only, for the sake of simplicity, the proof in the case of $\ell = 2$,

$$\begin{aligned}
\sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} &= \sum_{m\in\Omega} \left(\Delta_1^2 u_{m_1-1,m_2,n+1} + \Delta_2^2 u_{m_1,m_2-1,n+1}\right) \\
&= \sum_{m\in\Omega} \big(u_{M_1+1,m_2,n+1} - u_{M_1,m_2,n+1} - u_{1,m_2,n+1} + u_{0,m_2,n+1} \\
&\qquad + u_{m_1,M_2+1,n+1} - u_{m_1,M_2,n+1} - u_{m_1,1,n+1} + u_{m_1,0,n+1}\big) \\
&= \sum_{m\in\Omega} \big(\Delta_1 u_{m_1,m_2,n+1}\big|_{m_1=M_1} - \Delta_1 u_{m_1,m_2,n+1}\big|_{m_1=0} \\
&\qquad + \Delta_2 u_{m_1,m_2,n+1}\big|_{m_2=M_2} - \Delta_2 u_{m_1,m_2,n+1}\big|_{m_2=0}\big) \\
&= \sum_{m_2=1}^{M_2} \big(\Delta_1 u_{m_1,m_2,n+1}\big|_{m_1=M_1} - \Delta_1 u_{m_1,m_2,n+1}\big|_{m_1=0}\big) \\
&\quad + \sum_{m_1=1}^{M_1} \big(\Delta_2 u_{m_1,m_2,n+1}\big|_{m_2=M_2} - \Delta_2 u_{m_1,m_2,n+1}\big|_{m_2=0}\big).
\end{aligned} \tag{3.548}$$

Noting that the first term in the above is the sum of the normal differences on both left and right boundaries and the second one on both upper and lower boundaries of Ω, we have that the equality (3.547) holds and complete the proof. $\square$

3.8.2. Parabolic equations

Consider the nonlinear parabolic difference equations of neutral type of the form

$$\begin{aligned}\Delta_2\left(u_{m,n}-\sum_{k\in K}r_{k,n}u_{m,n-\alpha_k}\right)+p_{m,n}u_{m,n}+\sum_{i\in I}p_{m,n}^{(i)}f_i(u_{m,n-\beta_i})\\ =q_n\nabla^2u_{m-1,n+1}+\sum_{j\in J}q_{j,n}\nabla^2u_{m-1,n+1-\gamma_j}\quad\text{for } m\in\Omega,\ n\in N_{n_0},\end{aligned}\tag{3.549}$$

where $I\triangleq\{1,\dots,I_0\}$, $J\triangleq\{1,\dots,J_0\}$, $K\triangleq\{1,\dots,K_0\}$, Ω is a convex connected net.

We assume throughout this section that

(H_1) $q_n\in N_{n_0}\to R^+$ and $q_{j,n}\in J\times N_{n_0}\to R^+$;

(H_2) $p_{m,n}\in\Omega\times N_{n_0}\to R^+$, $p_{m,n}^{(i)}\in I\times\Omega\times N_{n_0}\to R^+$, $p_n\triangleq\min_{m\in\Omega}\{p_{m,n}\}$, $p_{i,n}\triangleq\min_{m\in\Omega}\{p_{m,n}^{(i)}\}$ for $i\in I$ and $n\in N_{n_0}$;

(H_3) $\alpha_k\in K\to N_1$, $\beta_i\in I\to N_1$ and $\gamma_j\in J\to N_1$;

(H_4) $f_i\in C(R,R)$ are convex and increasing on $R^+\setminus\{0\}$, $uf_i(u)>0$ for $u\neq0$ and $i\in I$ and $f(0)=0$;

(H_5) $r_{k,n}\in K\times N_{n_0}\to R^+$ and $\sum_{k\in K}r_{k,n}\leq1$.

Consider the initial boundary value problem (IBVP) (3.549) with the homogeneous Robin boundary condition (RBC)

$$\Delta_N u_{m-1,n}+g_{m,n}u_{m,n}=0\quad\text{on }\partial\Omega\times N_{n_0}\tag{3.550}$$

and the initial condition (IC)

$$u_{m,s}=\mu_{m,s}\quad\text{for } n_0-\tau\leq s\leq n_0,\tag{3.551}$$

where $\tau=\max\{\alpha_k,\beta_i,\gamma_j : k\in K,\ i\in I \text{ and } j\in J\}$ and $g_{m,n}\in\partial\Omega\times N_{n_0}\to R^+$.

By a solution of IBVP (3.549)–(3.551) we mean a sequence $\{u_{m,n}\}$ which satisfies (3.549) for $(m,n)\in\Omega\times N_{n_0}$, RBC (3.550) for $(m,n)\in\partial\Omega\times N_{n_0}$, and IC (3.551) for $(m,n)\in\Omega\times\{n_0-\tau,n_0-\tau+1,\dots,n_0\}$. Similar to Chapter 2, by the successive calculation, it is easy to show that IBVP (3.549)–(3.551) has a unique solution.

Our objection in this section is to present sufficient conditions which imply that every solution $\{u_{m,n}\}$ of IBVP (3.549)–(3.551) is oscillatory in $\Omega\times N_{n_0}$ in the sense that there are no solutions to be eventually positive or eventually negative in n.

Theorem 3.114. *Let hypotheses* (H_1)*–*(H_5) *hold. Suppose that there exist two positive constants* $B, C > 0$ *and an* $i_0 \in I$ *such that* $f_{i_0}(u)/u \geq C$ *for* $u \neq 0$ *and* $p_n, p_{i_0,n} \geq B$ *for* $n \in N_{n_0}$*. If*

$$\limsup_{n\to\infty} \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} > \frac{1}{C}, \tag{3.552}$$

then every solution $\{u_{m,n}\}$ *of IBVP* (3.549)–(3.551) *is oscillatory in* $\Omega \times N_{n_0}$*.*

Proof. Suppose that it is not true and $\{u_{m,n}\}$ is a nonoscillatory solution. Without loss of generality, we may assume that there exists an $n_1 \in N_{n_0}$ such that $u_{m,n} > 0$ for $n \in N_{n_1}$. Hence $u_{m,n-\alpha_k}$, $u_{m,n-\beta_i}$ and $u_{m,n-\gamma_j} > 0$ for $n \in N_{n_1+\tau} \triangleq N_{n_2}$.

Summing (3.549) over Ω, we have

$$\begin{aligned} &\Delta_2\left(\sum_{m\in\Omega} u_{m,n} - \sum_{k\in K} r_{k,n} \sum_{m\in\Omega} u_{m,n-\alpha_k}\right) + \sum_{m\in\Omega} p_{m,n}u_{m,n} + \sum_{i\in I}\sum_{m\in\Omega} p_{m,n}^{(i)} f_i(u_{m,n-\beta_i}) \\ &\quad = q_n \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1-\gamma_j} \quad \text{for } (m,n) \in \Omega\times N_{n_2}. \end{aligned} \tag{3.553}$$

From (H_4), Theorem 3.113, and the Jensen's inequality, it follows that

$$\begin{aligned} \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} &= \sum_{m\in\partial\Omega} \Delta_N u_{m-1,n+1} \\ &= -\sum_{m\in\partial\Omega} g_{m,n+1}u_{m,n+1} \leq 0 \quad \text{for } n \in N_{n_2}, \\ \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1-\gamma_j} &= \sum_{m\in\partial\Omega} \Delta_N u_{m-1,n+1-\gamma_j} \\ &= -\sum_{m\in\partial\Omega} g_{m,n+1-\gamma_j} u_{m,n+1-\gamma_j} \leq 0 \end{aligned} \tag{3.554}$$

for $j \in J$ and $n \in N_{n_2}$,

$$\sum_{m\in\Omega} p_{m,n}u_{m,n} \geq p_n \sum_{m\in\Omega} u_{m,n} = |\Omega| p_n v_n \quad \text{for } n \in N_{n_2}, \tag{3.555}$$

where $v_n = (1/|\Omega|)\sum_{m\in\Omega} u_{m,n}$ and $|\Omega|$ is the number of points in Ω, and

$$\sum_{m\in\Omega} p_{m,n}^{(i)} f_i(u_{m,n-\beta_i}) \geq p_{i,n} \sum_{m\in\Omega} f_i(u_{m,n-\beta_i}) \geq p_{i,n} f_i\left(\frac{1}{|\Omega|}\sum_{m\in\Omega} u_{m,n-\beta_i}\right)|\Omega| \tag{3.556}$$

for $i \in I$ and $n \in N_{n_2}$. Thus, we obtain by (3.553)–(3.556) that

$$\Delta\left(v_n - \sum_{k\in K} r_{k,n} v_{n-\alpha_k}\right) + \sum_{i\in I} p_{i,n} f_i\left(v_{n-\beta_i}\right) \le 0 \quad \text{for } n \in N_{n_2}, \tag{3.557}$$

where Δ is the ordinary difference operator.

Let

$$w_n = v_n - \sum_{k\in K} r_{k,n} v_{n-\alpha_k}. \tag{3.558}$$

We have by (H_5) and (3.557)

$$\Delta w_n < 0, \qquad w_n \le v_n. \tag{3.559}$$

This follows $\lim_{n\to\infty} w_n = L$. We can prove that $L > -\infty$. In fact, if $L = -\infty$, then v_n is unbounded. Hence, there exists an $n_3 \in N_{n_2}$ such that

$$w_{n_3} < 0, \qquad v_{n_3} = \max_{n_2 \le n \le n_3} v_n. \tag{3.560}$$

It then follows from (H_5) that

$$v_{n_3} - \sum_{k\in K} r_{k,n} v_{n_3-\alpha_k} \ge v_{n_3}\left(1 - \sum_{k\in K} r_{k,n}\right) \ge 0, \tag{3.561}$$

which contradicts (3.560). Hence, $L > -\infty$ and is finite.

Summing (3.557) from n_3 to n, we obtain

$$\begin{aligned} 0 < B\sum_{s=n_3}^{n} f_{i_0}\left(v_{s-\beta_{i_0}}\right) &\le \sum_{i\in I}\sum_{s=n_3}^{n} p_{i,s} f_i\left(v_{s-\beta_i}\right) \\ &\le -\sum_{s=n_3}^{n} \Delta w_s = w_{n_3} - w_{n+1} \le w_{n_3} - L < \infty. \end{aligned} \tag{3.562}$$

Therefore $f_{i_0}(v_{n-\beta_{i_0}})$ is summable and $\lim_{n\to\infty} v_n = 0$ by (H_4). It then follows that $\lim_{n\to\infty} w_n = 0$.

From (3.557) and (3.559), there exists an $n_4 \in N_{n_3}$ such that

$$\Delta w_n + \sum_{i\in I} p_{i,n} f_i\left(w_{n-\beta_i}\right) \le 0 \quad \text{for } n \in N_{n_4}. \tag{3.563}$$

Moreover,

$$\Delta w_n + p_{i_0,n} f_{i_0}\left(w_{n-\beta_{i_0}}\right) \le 0 \quad \text{for some } i_0 \in I,\ n \in N_{n_4}. \tag{3.564}$$

Summing (3.564) from $n-\beta_{i_0}$ to n, we have

$$w_{n+1} - w_{n-\beta_{i_0}} + \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} f_{i_0}\left(w_{s-\beta_{i_0}}\right) \leq 0 \quad \text{for } n \in N_{n_4}. \tag{3.565}$$

Since $\Delta w_n < 0$ and $f_{i_0}(u)$ is increasing on $R^+ \setminus \{0\}$, we have

$$\begin{gathered} w_{n+1} - w_{n-\beta_{i_0}} + f_{i_0}\left(w_{n-\beta_{i_0}}\right) \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} \leq 0 \quad \text{for } n \in N_{n_4}, \\ \frac{f_{i_0}\left(w_{n-\beta_{i_0}}\right)}{w_{n-\beta_{i_0}}} \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} \leq 1 - \frac{w_{n+1}}{w_{n-\beta_{i_0}}} < 1. \end{gathered} \tag{3.566}$$

Hence $C\sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} < 1$ and

$$\limsup_{n\to\infty} \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} \leq \frac{1}{C}, \tag{3.567}$$

which contradicts (3.552). This completes the proof. □

Theorem 3.115. *Let (H_1)–(H_5) hold. Suppose that there exist $C_i \geq 0$ and a $B > 0$ such that $f_i(u)/u \geq C_i$ for $u \neq 0$ and $p_{i_0,n} \geq B$ for some $i_0 \in I$. If*

$$\limsup_{n\to\infty} \sum_{s=n-\tau}^{n} \sum_{i\in I} C_i p_{i,s} > 1, \tag{3.568}$$

then every solution of IBVP (3.549)–(3.551) *is oscillatory in $\Omega \times N_{n_0}$.*

Proof. Let $\{u_{m,n}\}$ be a nonoscillatory solution of IBVP (3.549)–(3.551). Without loss of generality, we assume that $u_{m,n} > 0$ for some $n_5 \in N_{n_4}$. Hence, we have $u_{m,n-\alpha_k}, u_{m,n-\beta_i}$ and $u_{m,n-\gamma_j} > 0$ for $n \in N_{n_5+\tau} \triangleq N_{n_6}$. As in the proof of Theorem 3.114, we know that (3.557)–(3.563) hold. Summing (3.563) from $n-\beta$ to n, we have

$$w_{n+1} - w_{n-\beta} + \sum_{i\in I} \sum_{s=n-\beta}^{n} p_{i,s} f_i\left(w_{s-\beta_i}\right) \leq 0 \qquad \text{for } n \in N_{n_6}, \tag{3.569}$$

where $\beta \triangleq \max_{i\in I}\{\beta_i\}$ and n_6 is sufficiently large. From (3.560), we have

$$w_{n+1} - w_{n-\beta} + \sum_{i\in I} f_i(w_{n-\beta}) \sum_{s=n-\beta}^{n} p_{i,s} \le 0 \quad \text{for } n \in N_{n_6}. \tag{3.570}$$

It follows that

$$\sum_{i\in I} \frac{f_i(w_{n-\beta})}{w_{n-\beta}} \sum_{s=n-\beta}^{n} p_{i,s} \le 1 - \frac{w_{n+1}}{w_{n-\beta}} < 1 \quad \text{for } n \in N_{n_6} \tag{3.571}$$

and $\sum_{s=n-\beta}^{n}\sum_{i\in I} C_i p_{i,s} \le 1$, which contradicts (3.568). The proof is thus complete. □

Corollary 3.116. *Assume that* (H_1)*–*(H_5) *hold. If the difference inequality* (3.557) *(resp.,* (3.563)*) has no eventually positive solutions, then every solution* $\{u_{m,n}\}$ *of IBVP* (3.549)–(3.551) *is oscillatory in* $\Omega \times N_{n_0}$.

3.8.3. Hyperbolic equations

We consider the nonlinear hyperbolic partial difference equations of the form

$$\begin{aligned}\Delta_2\left[s_n\Delta_2\left(u_{m,n} + \sum_{k\in K} r_{k,n}u_{m,n-\alpha_k}\right)\right] + p_{m,n}u_{m,n} + \sum_{i\in I} p_{m,n}^{(i)} f_i(u_{m,n-\beta_i}) \\ = q_n\nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n}\nabla^2 u_{m-1,n+1-\gamma_j}, \quad (m,n)\in\Omega\times N_{n_0}\end{aligned} \tag{3.572}$$

with RBC (3.550) and IC (3.551).

We assume in this section that (H_1)–(H_5) hold and

(H_6) $s_n \in N_{n_0} \to R^+ \setminus \{0\}$ and $\sum_{n=n_0}^{\infty}(1/s_n) = \infty$.

Theorem 3.117. *Let* (H_1)*–*(H_6) *hold. Suppose that for any constant* $A > 0$, *there exists an* i_0 *such that*

$$\sum_{n=n_0}^{\infty} p_{i_0,n} f_{i_0}\left[A\left(1 - \sum_{k\in K} r_{k,n-\beta_{i_0}}\right)\right] = \infty. \tag{3.573}$$

Then every solution of IBVP (3.572), (3.550), *and* (3.551) *is oscillatory in* $\Omega \times N_{n_0}$.

Proof. Let $\{u_{m,n}\}$ be such a nonoscillatory solution of IBVP (3.572), (3.550), and (3.551) that $u_{m,n} > 0$ for some $n_1 \in N_{n_0}$ and $n \in N_{n_1}$. Then we have $u_{m,n-\alpha_k}$, $u_{m,n-\beta_i}$, and $u_{m,n-\gamma_j} > 0$ for $n \in N_{n_1+\tau} \triangleq N_{n_2}$, where $i \in I$, $j \in J$, and $k \in K$.

Summing (3.572) in the both sides over Ω, we have, for $(m,n) \in \Omega \times N_{n_2}$,

$$\Delta_2\left[s_n\Delta_2\left(\sum_{m\in\Omega} u_{m,n} + \sum_{k\in K} r_{k,n} \sum_{m\in\Omega} u_{m,n-\alpha_k}\right)\right]$$

$$+ \sum_{m\in\Omega} p_{m,n}u_{m,n} + \sum_{i\in I}\sum_{m\in\Omega} p^{(i)}_{m,n} f_i\left(u_{m,n-\beta_i}\right) \tag{3.574}$$

$$= q_n \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1-\gamma_j}.$$

As in Theorem 3.114, (3.554)–(3.556) hold. Therefore, we obtain

$$\Delta\left[s_n\Delta\left(v_n + \sum_{k\in K} r_{k,n}v_{n-\alpha_k}\right)\right] + \sum_{i\in I} p_{i,n} f_i\left(v_{n-\beta_i}\right) \le 0 \quad \text{for } n \in N_{n_2}. \tag{3.575}$$

Let

$$w_n = v_n + \sum_{k\in K} r_{k,n}v_{n-\alpha_k}. \tag{3.576}$$

Then we have

$$w_n > 0, \quad w_n \ge v_n \quad \text{for } n \in N_{n_2}. \tag{3.577}$$

From (H_2), (H_3), and (3.575), we obtain

$$\Delta\left(s_n\Delta w_n\right) \le -\sum_{i\in I} p_{i,n} f_i\left(v_{n-\beta_i}\right) \le 0 \quad \text{for } n \in N_{n_2}, \tag{3.578}$$

which means that $\{s_n\Delta w_n\}$ is decreasing. We claim that

$$s_n\Delta w_n \ge 0 \quad \text{for } n \in N_{n_2}. \tag{3.579}$$

Consequently,

$$\Delta w_n \ge 0 \quad \text{for } n \in N_{n_2}. \tag{3.580}$$

If it is not true, then there exists an $n_3 \in N_{n_2}$ such that $s_{n_3}\Delta w_{n_3} < 0$ and $s_n\Delta w_n \ge 0$ for $n_2 \le n < n_3$. Using (3.578), we have $\Delta w_n \le (1/s_n)s_{n_3}\Delta w_{n_3}$ for $n \in N_{n_3}$, which follows that

$$w_{n+1} - w_{n_3} \le s_{n_3}\Delta w_{n_3} \sum_{n=n_3}^{n} \frac{1}{s_n} \quad \text{for } n \in N_{n_3}. \tag{3.581}$$

Then we have $w_n < 0$ as $n \to \infty$, which contradicts (3.577).

We know from (3.578) that for some $i_0 \in I$ we have

$$\Delta(s_n \Delta w_n) + p_{i_0,n} f_{i_0}(v_{n-\beta_{i_0}}) \leq 0 \quad \text{for } n \in N_{n_3}, \tag{3.582}$$

which follows

$$\Delta(s_n \Delta w_n) + p_{i_0,n} f_{i_0}\left(w_{n-\beta_{i_0}} - \sum_{k \in K} r_{k,n-\beta_{i_0}} v_{n-\beta_{i_0}-\alpha_k}\right) \leq 0 \quad \text{for } n \in N_{n_3}. \tag{3.583}$$

From (3.577), (3.580), and (3.583), we have

$$\Delta(s_n \Delta w_n) + p_{i_0,n} f_{i_0}\left[w_{n-\beta_{i_0}}\left(1 - \sum_{k \in K} r_{k,n-\beta_{i_0}}\right)\right] \leq 0 \quad \text{for } n \in N_{n_3}. \tag{3.584}$$

Summing (3.584) from n_3 to n and using (3.580), we have

$$s_{n+1}\Delta w_{n+1} - s_{n_3}\Delta w_{n_3} + \sum_{t=n_3}^{n} p_{i_0,t} f_{i_0}\left[w_{n_3-\beta_{i_0}}\left(1 - \sum_{k \in K} r_{k,t-\beta_{i_0}}\right)\right] \leq 0. \tag{3.585}$$

By (3.578) and (3.579), letting $n \to \infty$ in (3.585), we have

$$\sum_{t=n_3}^{\infty} p_{i_0,t} f_{i_0}\left[w_{n_3-\beta_{i_0}}\left(1 - \sum_{k \in K} r_{k,t-\beta_{i_0}}\right)\right] < \infty. \tag{3.586}$$

Let $A = w_{n_3-\beta_{i_0}}$. Then we have

$$\sum_{t=n_3}^{\infty} p_{i_0,t} f_{i_0}\left[A\left(1 - \sum_{k \in K} r_{k,t-\beta_{i_0}}\right)\right] < \infty, \tag{3.587}$$

which contradicts (3.573). Thus, this completes the proof. □

Remark 3.118. From the proof of Theorem 3.115, if the second-order delay difference inequality

$$\Delta(s_n \Delta w_n) + C p_{i_0}\left(1 - \sum_{k \in K} r_{k,n-\beta_{i_0}}\right) w_{n-\beta_{i_0}} \leq 0, \quad n \geq n_3 \tag{3.588}$$

has no positive solutions, then the conclusion of Theorem 3.115 holds. Hence the well known Ricatti technique can be used to derive some oscillation criteria for the oscillation of IBVP (3.572), (3.550), and (3.551).

Example 3.119. Consider the parabolic equation

$$\Delta_2\left(u_{m,n}-\frac{1}{2}u_{m,n-1}\right)+u_{m,n}+2e^{-m^2}u_{m,n-i}e^{u^2_{m,n-i}}$$
$$=q_n\nabla^2 u_{m-1,n+1}+\sum_{j\in J}q_{j,n}\nabla^2 u_{m-1,n+1-\gamma_j}\quad \text{for } m=1,2,\ldots,M,\ n\in N_{n_0}, \tag{3.589}$$

where $i>[(1/2)e^{M^2}-1]$, ($[\cdot]$ is the integer function) is an even integer, q_n, $q_{j,n}$, and γ_j satisfy the hypotheses in Theorem 3.114.

We have $r_n = 1/2 < 1$, $p_{m,n} = 1 > B$, $p^*_{m,n} = 2e^{-m^2} \geq B$, where $B \triangleq \min\{1,2e^{-M^2}\}$, $f(u)=ue^{u^2}$, $f(u)/u=e^{u^2}\geq 1\triangleq C$, and

$$\sum_{s=n-i}^{n}p_{*,s}=(i+1)2e^{-M^2}>\frac{1}{C}=1. \tag{3.590}$$

By Theorem 3.114, every solution of (3.589) is oscillatory. In fact, $u_{m,n}=(-1)^n m$ is an oscillatory solution of (3.589).

Example 3.120. Consider the hyperbolic equation

$$\Delta_2\left[n\Delta_2\left(u_{m,n}+\frac{1}{2}u_{m,n-1}\right)\right]+3u_{m,n}+\frac{5n^3-18n^2+10n+12}{2(n-1)n(n+1)(n+2)}u_{m,n-2}$$
$$=q_n\nabla^2 u_{m-1,n+1}+\sum_{j\in J}q_{j,n}\nabla^2 u_{m-1,n+1-\gamma_j}\quad \text{for } m=1,\ldots,M,\ n\in N_{n_0}. \tag{3.591}$$

It is easy to see that the conditions in Theorem 3.115 are all satisfied. Then every solution of (3.591) is oscillatory. In fact,

$$u_{m,n}=\frac{(-1)^n m}{n} \tag{3.592}$$

is an oscillatory solution of (3.591).

Example 3.121. Consider the hyperbolic equation

$$\Delta_2\left[n\Delta_2\left(u_{m,n}+\frac{n-1}{n}u_{m,n-1}\right)\right]+\frac{1}{2}u_{m,n}$$
$$+m^{1/3}(n-1)^{4/3}\frac{n^5-11n^4-23n^3-9n^2+8n+4}{2(n-1)^2n^2(n+1)^2(n+2)^2}u^{2/3}_{m,n-1} \tag{3.593}$$
$$=q_n\nabla^2 u_{m-1,n+1}+\sum_{j\in J}q_{j,n}\nabla^2 u_{m-1,n+1-\gamma_j},\quad m=1,\ldots,M,\ n\in N_{n_0}.$$

It is easy for one to see that (3.573) is false this time. In fact, (3.593) has a nonoscillatory solution $u_{m,n} = m/n^2$.

3.9. Notes

The material of Section 3.2.1 is taken from Zhang and Yu [186]. The linearized oscillation theorem for the partial difference equation (3.31) can be seen from Zhang and Xing [184]. The results in Section 3.2.2 are adopted from Zhang [162]. The material of Section 3.2.3 is taken from Liu and Zhang [97]. The results in Section 3.3.1 are taken from Zhang and Liu [173, 175]. The material of Section 3.3.2 is taken from Zhang and Liu [168]. The material of Section 3.3.3 is adopted from Zhang and Xing [183]. The material of Section 3.4 is taken from Zhang and Xing [181]. The results in Section 3.5.1 are taken from Zhang et al. [190]. The contents of Section 3.5.2 are taken from Zhou [192], Xing and Zhang [159], respectively. The material of Section 3.6 is adopted from Zhang and Saker [177]. The material in Section 3.7.1 is taken from Cheng and Zhang [42]. The material of Section 3.7.2 is taken from Cheng et al. [40]. The material of Section 3.8 is taken from Shi et al. [126].

4 Stability of delay partial difference equations

4.1. Introduction

In this chapter, we consider the stability of delay partial difference equations. It is well known that the conditions of the global attractivity of the trivial solution of the ordinary difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \dots, \tag{4.1}$$

were obtained in [58, 179].

Consider the delay partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + P_{m,n} A_{m-k,n-l} = 0, \tag{4.2}$$

where $\{P_{m,n}\}_{m,n=0}^{\infty}$ is a real double sequence, k, l are nonnegative integers.

Let $\Omega = N_{-k} \times N_{-l} \setminus N_1 \times N_0$ be an initial value set

$$A_{i,j} = \varphi_{i,j}, \quad (i, j) \in \Omega, \tag{4.3}$$

where $\varphi_{i,j}$ is a given initial function.

The sequence $\{A_{i,j}\}$ is called the solution of the initial value problem (4.2) and (4.3) if it satisfies (4.2) and (4.3). The (trivial) solution of (4.2) is said to be global attractive if, for any given initial function $\{\varphi_{i,j}\}$, the corresponding solution $\{A_{i,j}\}$ satisfies $\lim_{i,j\to\infty} A_{i,j} = 0$.

The first question is that if (4.2) has the global attractivity, which is similar to the ordinary difference equations mentioned.

In the following, we use the triangle graphical method and the induction method to prove that, for any double sequence $\{P_{m,n}\}_{m,n=0}^{\infty}$, the trivial solution of (4.2) is not globally attractive, that is, we can always construct a solution of (4.2) which does not converge to zero.

We first consider a special case of (4.2):

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} = 0, \quad m, n = 0, 1, 2, \dots. \tag{4.4}$$

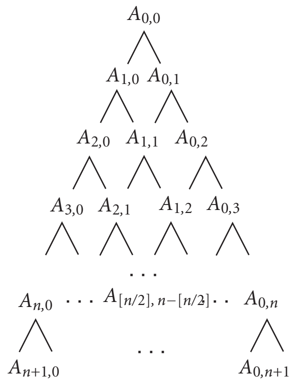


Figure 4.1

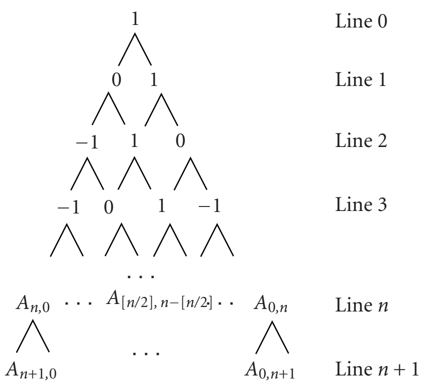


Figure 4.2

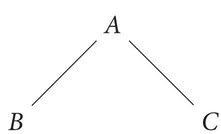
Theorem 4.1. *The trivial solution of* (4.4) *is not globally attractive.*

Proof. Let $A_{0,0}$=1, from (4.4), we have

$$A_{m,n} = A_{m+1,n} + A_{m,n+1}, \quad m,n = 0,1,2,\dots. \tag{4.5}$$

From (4.5), we can make the triangle graphs (see Figures 4.1 and 4.2).

Where $[\cdot]$ denotes the largest integer function, the triangle



satisfies $A = B + C$ and the corresponding numbers in the two graphs are equal.

We will prove the theorem by induction.

From Figure 4.2, we see that if $A_{0,0} = 1$, then each number in line 1 exists, for example, we can select 0 and 1. Suppose each number in line n exists and satisfies $A_{[n/2],\,n-[n/2]} = 1$, that is, there is a number equaling 1 in the middle of the line (e.g., line 2) and if there are two numbers in the middle of the line (e.g., line 3), we can choose the number on the right side of the middle of the line to be 1.

Next we will prove that each number in line $n+1$ exists and there is a number, which equals 1 in the middle of the line.

When n is even, let

$$A_{[(n+1)/2],\,n+1-[(n+1)/2]} = A_{[n/2],\,n-[n/2]+1} = 1. \tag{4.6}$$

Since there is only a number which is independent in line $n+1$, the equality (4.6) can be regarded as the initial value. From (4.4), we get

$$\begin{aligned}
A_{[(n+1)/2]+1,\,n-[(n+1)/2]} &= A_{[n/2]+1,\,n-[n/2]} = A_{[n/2],\,n-[n/2]} - A_{[n/2],\,n-[n/2]+1} \\
&= A_{[n/2],\,n-[n/2]} - 1, \\
A_{[(n+1)/2]+2,\,n-[(n+1)/2]-1} &= A_{[n/2]+2,\,n-[n/2]-1} = A_{[n/2]+1,n-[n/2]-1} - A_{[n/2]+1,n-[n/2]}, \\
&\vdots \\
A_{n+1,0} &= A_{n,0} - A_{n,1}, \\
A_{[(n+1)/2]-1,n-[(n+1)/2]+2} &= A_{[n/2]-1,n-[n/2]+2} = A_{[n/2]-1,n-[n/2]+1} - A_{[n/2],\,n-[n/2]+1} \\
&= A_{[n/2]-1,n-[n/2]+1} - 1, \\
&\vdots \\
A_{1,n} &= A_{1,n-1} - A_{2,n-1}, \\
A_{0,n+1} &= A_{0,n} - A_{1,n}.
\end{aligned} \tag{4.7}$$

In the above equalities, we see that the first term of the right side is just the corresponding value of line n and the second term is known by recurrence, so each number of the left side exists and they are just the corresponding values of line $n+1$ and there is a number, which equals 1 in the middle.

When n is odd, similar to the above proof, we can obtain the same result.

Summarizing the above discussion, for any natural number n, each number of line n can be confirmed by the triangle graphical method and there is a number, which equals 1 in the middle, let

$$\varphi_{0,0} = 1, \varphi_{0,1} = 1, \varphi_{0,2} = 0, \varphi_{0,3} = -1, \ldots, \tag{4.8}$$

then by the triangle graphical method we can confirm a double sequence $\{A_{m,n}\}$ which is a solution of (4.4) and satisfies the initial condition

$$A_{0,j} = \varphi_{0,j}, \quad j = 0, 1, 2, \ldots. \tag{4.9}$$

Obviously, $\{A_{m,n}\}$ does not converge to zero when m, n converges to ∞, respectively. The proof is completed. □

Theorem 4.2. *If l and k are nonnegative integers and are not equal zero at the same time, then the trivial solution of* (4.2) *is not globally attractive.*

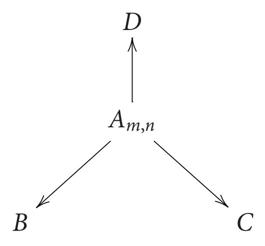
Proof. Let $A_{0,0} = 1$, assume $\{\varphi_{i,j}\}$ is an any given real sequence defined on the free initial value set Ω_2, let

$$A_{m,n} = \varphi_{m,n}, \quad (m, n) \in \Omega_2. \tag{4.10}$$

From (4.2), we have

$$A_{m,n} = A_{m+1,n} + A_{m,n+1} + P_{m,n}A_{m-k,n-l}. \tag{4.11}$$

From (4.11), we can make a triangle graph (see Figure 4.3), where



satisfies $A_{m,n} = B + C + P_{m,n}D$.

From Figure 4.3, we see that each number in line $(1, 1)$ exists or is known, so we choose an initial function such that $A_{0,1} = 1$, that is, there is a number, which is 1 in the middle of line $(1, 1)$.

Suppose each number exists or is known upon line $(1, n)$ and there is a number which is 1 in the middle of line $(1, n)$, that is, $A_{[n/2],\, n-[n/2]} = 1$, next we will prove that each number exists in line $(1, n + 1)$ and line $(0, n + 1)$ and there is a number, which is 1 in the middle of line $(1, n + 1)$.

When n is odd, let

$$A_{[(n+1)/2],\, n+1-[(n+1)/2]} = A_{[n/2]+1,\, n-[n/2]} = 1. \tag{4.12}$$

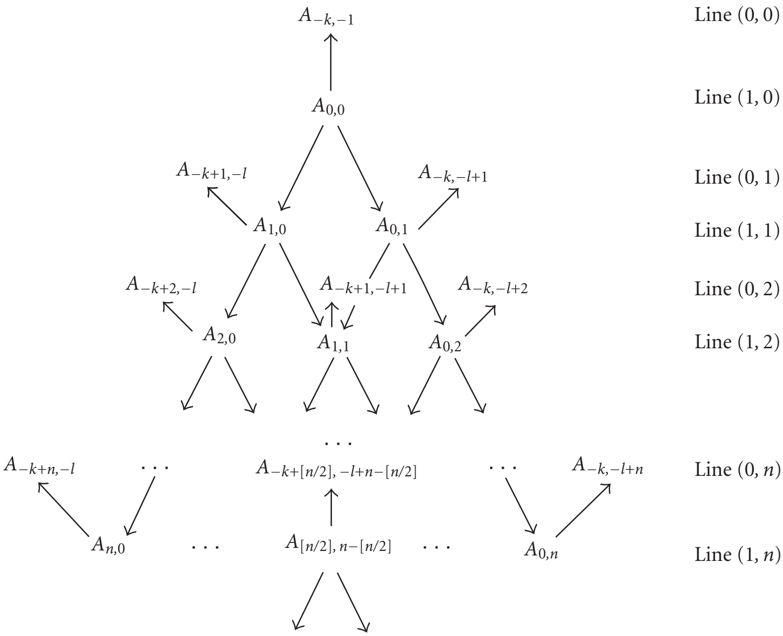


Figure 4.3

From (4.11), we have

$$
\begin{aligned}
A_{[(n+1)/2]+1,\,n-[(n+1)/2]} &= A_{[n/2]+2,\,n-[n/2]-1}\\
&= A_{[n/2]+1,n-[n/2]-1} - A_{[n/2]+1,\,n-[n/2]}\\
&\quad - P_{[n/2]+1,n-[n/2]-1}A_{-k+[n/2]+1,-l+n-[n/2]-1}\\
&= A_{[n/2]+1,n-[n/2]-1} - 1 - P_{[n/2]+1,n-[n/2]-1}A_{-k+[n/2]+1,-l+n-[n/2]-1},\\
&\ \ \vdots\\
A_{n,1} &= A_{n-1,1} - A_{n-1,2} - P_{n-1,1}A_{-k+n-1,-l+1},\\
A_{n+1,0} &= A_{n,0} - A_{n,1} - P_{n,0}A_{-k+n,-l},\\
A_{[(n+1)/2]-1,n-[(n+1)/2]+2} &\\
&= A_{[n/2],\,n-[n/2]+1} = A_{[n/2],\,n-[n/2]} - A_{[n/2]+1,n-[n/2]}\\
&\quad - P_{[n/2],\,n-[n/2]}A_{-k+[n/2],\,-l+n-[n/2]}\\
&= A_{[n/2],\,n-[n/2]} - 1 - P_{[n/2],\,n-[n/2]}A_{-k+[n/2],\,-l+n-[n/2]},\\
&\ \ \vdots\\
A_{1,n} &= A_{1,n-1} - A_{2,n-1} - P_{1,n-1}A_{-k+1,-l+n-1},\\
A_{0,n+1} &= A_{0,n} - A_{1,n} - P_{0,n}A_{-k,-l+n}.
\end{aligned}
\tag{4.13}
$$

In the above equalities, the first term on the right side is just the corresponding number in line $(1, n)$ and is known, the second term is also known by recurrence, the first factor of the third term is a coefficient and is also known, the second factor of the third term is some number of every line upon line $(1, n)$, so each number in line $(1, n+1)$ exists and the middle number is 1 (see (4.12)).

When n is even, similar to the above proof, we can obtain the same result.

Summarizing the above discussion, we can construct a solution $\{A_{m,n}\}$ of (4.2) by the triangle graphical method such that there exists a subsequence $\{m_r, n_r\}$ such that $m_r \to \infty$, $n_r \to \infty$ when $r \to \infty$ and $\lim_{r\to\infty} A_{m_r,n_r} = 1$. Hence the trivial solution of (4.2) is not globally attractive. The proof is completed. □

The above result shows that there exists great difference between the partial difference equation (4.2) and the corresponding ordinary difference equation mentioned in the global attractivity. Therefore, in this chapter, we mainly consider the local stability of the delay partial difference equations. In Section 4.2, we consider the stability and instability of scalar PDEs. In Section 4.3, the stability of the linear PDE systems is studied. In Section 4.4, the stability of some discrete delay logistic equations will be considered. In Section 4.5, we present a result for the L^2 stability of a class of the initial boundary value problem. In Section 4.6, we consider the stability of the reaction diffusion systems.

4.2. Stability criteria of delay partial difference equations

4.2.1. Stability of linear delay PDEs

Consider the delay partial difference equation

$$u_{i,j+1} = a_{i,j}u_{i+1,j} + b_{i,j}u_{i,j} + p_{i,j}u_{i-\sigma,j-\tau}, \tag{4.14}$$

where σ and τ are nonnegative integers, and $\{a_{i,j}\}$, $\{b_{i,j}\}$, and $\{p_{i,j}\}$ are real sequences defined on $i \ge 0$, $j \ge 0$.

By a solution of (4.14) we mean a real double sequence $\{u_{i,j}\}$ which is defined for $i \ge -\sigma$, and $j \ge -\tau$, and satisfies (4.14) for $i \ge 0$, $j \ge 0$.

Set $\Omega = N_{-\sigma} \times N_{-\tau} \backslash N_0 \times N_1$. Let the initial function φ be given on Ω. Obviously, the solution of the initial value problem of (4.14) is unique.

Let

$$\|\varphi\| = \sup_{(i,j)\in\Omega} |\varphi_{i,j}|. \tag{4.15}$$

For any positive real number $H > 0$, let $S_H = \{\varphi \mid \|\varphi\| < H\}$.

Definition 4.3. Equation (4.14) is said to be stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $\varphi \in S_\delta$, the corresponding solution $u = \{u_{i,j}\}$ of (4.14) satisfies

$$|u_{i,j}| < \varepsilon, \quad i, j \in N_0. \tag{4.16}$$

Definition 4.4. Equation (4.14) is said to be linearly stable if there exists an $M \geq 0$ such that every solution $u = \{u_{i,j}\}$ of (4.14) satisfies

$$|u_{i,j}| \leq M\|\varphi\|, \quad i, j \in N_0. \tag{4.17}$$

Obviously, (4.14) is linearly stable which implies that it is stable.

Definition 4.5. Equation (4.14) is said to be exponentially asymptotically stable if, for any $\delta > 0$, there exist a constant M_δ and a real number $\xi \in (0,1)$ such that $\varphi \in S_\delta$ implies that

$$|u_{i,j}| \leq M_\delta \xi^j \quad \text{or} \quad |u_{i,j}| \leq M_\delta \xi^i, \quad i, j \in N_0. \tag{4.18}$$

More general, we will adopt the following definition of the exponential asymptotic stability.

Definition 4.6. Equation (4.14) is said to be strongly exponentially asymptotically stable if, for any $\delta > 0$, there exist a constant M_δ and two real numbers $\xi, \eta \in (0,1)$ such that $\varphi \in S_\delta$ implies that

$$|u_{i,j}| \leq M_\delta \xi^i \eta^j, \quad i, j \in N_0. \tag{4.19}$$

Let $V(u,i,j) : R \times N_0^2 \to R^+ = [0,\infty)$. If for any solution $\{u_{i,j}\}$ of (4.14), there exists a constant $c > 0$ such that

$$V(u,i,j) \geq c|u_{i,j}|, \quad (i,j) \in N_0^2, \tag{4.20}$$

then $V(u,i,j)$ is said to be a positive Liapunov function.

The following lemma is obvious.

Lemma 4.7. *If there exist a positive Liapunov function $V(u,i,j)$ and a constant $M > 0$ such that*

$$V(u,i,j) \leq M\|\varphi\|, \quad i, j \in N_0, \tag{4.21}$$

where $\{u_{i,j}\}$ is a solution of (4.14) *with the initial function $\{\varphi_{i,j}\}$, then* (4.14) *is linearly stable (and hence stable).*

Let $A_{i,j} = |a_{i,j}| + |b_{i,j}| + |p_{i,j}|$ for any $i, j \in N_0$, and

$$\overline{a}_{i,j+1} = a_{i,j+1}A_{i+1,j}, \qquad \overline{b}_{i,j+1} = b_{i,j+1}A_{i,j}. \tag{4.22}$$

Theorem 4.8. *Assume that there exists a constant $C > 1$ such that*

$$\begin{gathered} |a_{i,0}| + |b_{i,0}| + |p_{i,0}| \leq C, \quad i \in N_0, \\ |\overline{a}_{i,j}| + |\overline{b}_{i,j}| + |p_{i,j}| \leq 1, \quad i \in N_0,\ j \geq 1. \end{gathered} \tag{4.23}$$

Then (4.14) *is linearly stable.*

Proof. For a given solution $\{u_{i,j}\}$ of (4.14), let

$$V(u,i,j)=\max_{i\geq 0}|u_{i,j}|,\quad j\geq 0,\quad w_u(j)=V(u,i,j). \tag{4.24}$$

From (4.14), we have

$$\begin{aligned}|u_{i,1}| &\leq |a_{i,0}||u_{i+1,0}|+|b_{i,0}||u_{i,0}|+|p_{i,0}||u_{i-\sigma,-\tau}|\\ &\leq (|a_{i,0}|+|b_{i,0}|+|p_{i,0}|)\|\varphi\|\leq C\|\varphi\|.\end{aligned} \tag{4.25}$$

Hence $w_u(1)\leq C\|\varphi\|$. Therefore,

$$\begin{aligned}|u_{i,2}| &\leq |a_{i,1}||u_{i+1,1}|+|b_{i,1}||u_{i,1}|+|p_{i,1}||u_{i-\sigma,1-\tau}|\\ &\leq |a_{i,1}|(|a_{i+1,0}||u_{i+2,0}|+|b_{i+1,0}||u_{i+1,0}|+|p_{i+1,0}||u_{i+1-\sigma,-\tau}|)\\ &\quad+|b_{i,1}|(|a_{i,0}||u_{i+1,0}|+|b_{i,0}||u_{i,0}|+|p_{i,0}||u_{i-\sigma,-\tau}|)\\ &\quad+|p_{i,1}||u_{i-\sigma,1-\tau}|\\ &\leq (|\overline{a}_{i,1}|+|\overline{b}_{i,1}|+|p_{i,1}|)\cdot C\|\varphi\|.\end{aligned} \tag{4.26}$$

Thus $w_u(2)\leq C\|\varphi\|$.

Assume that for some fixed integer $n>1$,

$$w_u(j)\leq C\|\varphi\|,\quad j\leq n. \tag{4.27}$$

Then we can obtain

$$\begin{aligned}|u_{i,n+1}| &\leq |a_{i,n}||u_{i+1,n}|+|b_{i,n}||u_{i,n}|+|p_{i,n}||u_{i-\sigma,n-\tau}|\\ &\leq (|\overline{a}_{i,n}|+|\overline{b}_{i,n}|+|p_{i,n}|)\cdot C\|\varphi\|\leq C\|\varphi\|.\end{aligned} \tag{4.28}$$

By induction, $w_u(n+1)\leq C\|\varphi\|$ for $n\geq 0$. Hence

$$|u_{i,j}|\leq w_u(j)\leq C\|\varphi\|,\quad (i,j)\in N_0^2. \tag{4.29}$$

The proof is completed. □

Example 4.9. Consider the partial difference equation

$$u_{i,j+1}=a_{i,j}u_{i+1,j}+b_{i,j}u_{i,j}+p_{i,j}u_{i-1,j-1}, \tag{4.30}$$

where

$$a_{i,j}=\frac{1}{2}+(-1)^i\frac{1}{2},\quad b_{i,j}=\frac{1}{8},\quad p_{i,j}=\frac{1}{8},\quad (i,j)\in N_0^2. \tag{4.31}$$

It is easy to see that $|a_{2i,j}| + |b_{2i,j}| + |p_{2i,j}| = 1.25 > 1$ for any $i \in N_0$. It is easy to obtain

$$\begin{aligned} A_{i,j} &= \frac{3}{4} + \frac{(-1)^i}{2}, \quad i, j \in N_0, \\ |a_{i,0}| + |b_{i,0}| + |p_{i,0}| &= \frac{3}{4} + \frac{(-1)^i}{2} \le 2 = C, \quad i \in N_0, \\ \overline{a}_{i,j+1} = a_{i,j+1}A_{i+1,j} &= \frac{1}{8} + \frac{(-1)^i}{8}, \\ \overline{b}_{i,j+1} = b_{i,j+1}A_{i,j} &= \frac{3}{32} + (-1)^i \frac{1}{16}. \end{aligned} \tag{4.32}$$

Then

$$|\overline{a}_{i,j+1}| + |\overline{b}_{i,j+1}| + |p_{i,j+1}| = \frac{11}{32} + (-1)^i \frac{3}{16}, \quad i \in N_0,\ j \ge 0. \tag{4.33}$$

By Theorem 4.8, we can conclude that (4.30) is linearly stable.

If (4.23) does not hold, then we can obtain the following result.

Theorem 4.10. *Let*

$$\begin{aligned} \overline{d}_0 &= \max_{i \ge 0} \{ |a_{i,0}| + |b_{i,0}| + |p_{i,0}| \}, \\ \overline{d}_j &= \max_{i \ge 0} \{ |\overline{a}_{i,j}| + |\overline{b}_{i,j}| + |p_{i,j}| \}, \quad j \ge 1, \end{aligned} \tag{4.34}$$

and $d_j = \max(1, \overline{d}_j) = 1 + r_j$ *for* $j \ge 0$. *If*

$$\sum_{j=0}^{\infty} r_j < \infty, \quad r_j \ge 0, \tag{4.35}$$

then (4.14) *is linearly stable.*

Proof. For a given solution $\{u_{i,j}\}$ of (4.14), let $w_u(j)$ be defined in (4.24) and

$$\overline{w}_u(j) = \max_{i \ge -\sigma} |u_{i,j}|, \quad j \ge -\tau. \tag{4.36}$$

It is easy to obtain $w_u(j) \le \overline{w}_u(j)$ for any $j \ge 0$ and

$$\overline{w}_u(j) \le \|\varphi\|, \quad -\tau \le j \le 0. \tag{4.37}$$

From (4.14), we have

$$\begin{aligned} |u_{i,1}| &\le |a_{i,0}| |u_{i+1,0}| + |b_{i,0}| |u_{i,0}| + |p_{i,0}| |u_{i-\sigma,-\tau}| \\ &\le (|a_{i,0}| + |b_{i,0}| + |p_{i,0}|) \cdot \max \{\overline{w}_u(0), \overline{w}_u(-\tau)\}. \end{aligned} \tag{4.38}$$

Hence in view of $d_j \geq 1$ for any $j \geq 0$, $\overline{w}_u(1) \leq d_0\|\varphi\|$. Similar to the proof of Theorem 4.8, we have

$$|u_{i,2}| \leq \left(|\overline{a}_{i,1}| + |\overline{b}_{i,1}| + |p_{i,1}|\right) \cdot \max\{\overline{w}_u(0), \overline{w}_u(1-\tau), \overline{w}_u(-\tau)\}. \tag{4.39}$$

Hence $\overline{w}_u(2) \leq d_1 d_0\|\varphi\|$.

Assume that for some fixed integer $n > 1$,

$$\overline{w}_u(j) \leq \prod_{i=0}^{j-1} d_i\|\varphi\| \quad \text{for } j \leq n. \tag{4.40}$$

Then we can obtain

$$\begin{aligned}
|u_{i,n+1}| \leq{} & |a_{i,n}||u_{i+1,n}| + |b_{i,n}||u_{i,n}| + |p_{i,n}||u_{i-\sigma,n-\tau}| \\
\leq{} & |a_{i,n}|\big(|a_{i+1,n-1}||u(i+2,n-1)| + |b_{i+1,n-1}||u_{i+1,n-1}| \\
& \qquad + |p_{i+1,n-1}||u_{i+1-\sigma,n-1-\tau}|\big) \\
& + |b_{i,n}|\big(|a_{i,n-1}||u_{i+1,n-1}| + |b_{i,n-1}||u_{i,n-1}| + |p_{i,n-1}||u_{i-\sigma,n-1-\tau}|\big) \\
& + |p_{i,n}||u_{i-\sigma,n-\tau}| \\
\leq{} & \left(|\overline{a}_{i,n}| + |\overline{b}_{i,n}| + |p_{i,n}|\right) \cdot \max\{\overline{w}_u(n-1), \overline{w}_u(n-\tau), \overline{w}_u(n-1-\tau)\}.
\end{aligned} \tag{4.41}$$

Hence by induction, $\overline{w}_u(n) \leq \prod_{i=0}^{n-1} d_i\|\varphi\|$ for $n \geq 0$. Thus

$$\begin{aligned}
\ln \overline{w}_u(n) &\leq \ln\|\varphi\| + \sum_{j=0}^{n-1} \ln d_j = \ln\|\varphi\| + \sum_{j=0}^{n-1} \ln\left(1 + r_j\right) \\
&\leq \ln\|\varphi\| + \sum_{j=0}^{n-1} r_j \leq \ln\|\varphi\| + \sum_{j=0}^{\infty} r_j,
\end{aligned} \tag{4.42}$$

and hence,

$$\overline{w}_u(n) \leq \|\varphi\| \exp\left(\sum_{j=0}^{\infty} r_j\right) = M\|\varphi\|, \tag{4.43}$$

where $M = \exp(\sum_{j=0}^{\infty} r_j)$. The proof is completed. □

Let $\widehat{A}_{i,j} = |a_{i,j}| + |b_{i,j}| + \xi^{-\tau}|p_{i,j}|$ for any $i, j \in N_0$ and

$$\widehat{a}_{i,j+1} = a_{i,j+1}\widehat{A}_{i+1,j}, \qquad \widehat{b}_{i,j+1} = b_{i,j+1}\widehat{A}_{i,j}. \tag{4.44}$$

Theorem 4.11. *Assume that* $\sigma = 0$, $\tau > 0$, *and there exist a constant* $C > 1$ *and a constant* $\xi \in (0,1)$ *such that*

$$\begin{gathered} |a_{i,0}| + |b_{i,0}| + |p_{i,0}| \le C, \quad i \in N_0, \\ |\hat{a}_{i,j}| + |\hat{b}_{i,j}| + \xi^{-\tau+1}|p_{i,j}| \le \xi^2, \quad i \in N_0,\ j \ge 1, \end{gathered} \tag{4.45}$$

then (4.14) *is exponentially asymptotically stable.*

Proof. Let $V(u,i,j)$ and $w_u(j)$ be defined in (4.24), then for any $\delta > 0$ and $\varphi \in S_\delta$, there exists a constant $M_\delta \ge C\xi^{-1}\|\varphi\| > 0$ such that

$$\begin{aligned} |u_{i,1}| &\le |a_{i,0}||u_{i+1,0}| + |b_{i,0}||u_{i,0}| + |p_{i,0}||u_{i-\sigma,-\tau}| \\ &\le (|a_{i,0}| + |b_{i,0}| + |p_{i,0}|)\|\varphi\| \le M_\delta \xi. \end{aligned} \tag{4.46}$$

Hence $w_u(1) \le M_\delta \xi$. Therefore,

$$\begin{aligned} |u_{i,2}| &\le |a_{i,1}||u_{i+1,1}| + |b_{i,1}||u_{i,1}| + |p_{i,1}||u_{i-\sigma,1-\tau}| \\ &\le |a_{i,1}|(|a_{i+1,0}||u_{i+2,0}| + |b_{i+1,0}||u_{i+1,0}| + |p_{i+1,0}||u_{i+1-\sigma,-\tau}|) \\ &\quad + |b_{i,1}|(|a_{i,0}||u_{i+1,0}| + |b_{i,0}||u_{i,0}| + |p_{i,0}||u_{i-\sigma,-\tau}|) + |p_{i,1}||u_{i,1-\tau}| \\ &\le (|\hat{a}_{i,1}| + |\hat{b}_{i,1}| + |p_{i,1}|)\|\varphi\| \le M_\delta \xi^2. \end{aligned} \tag{4.47}$$

Hence $w_u(2) \le M_\delta \xi^2$. In general, we have

$$w_u(j) \le M_\delta \xi^j, \quad i \in N_0,\ 0 \le j \le \tau. \tag{4.48}$$

Assume that for some fixed integer $n \ge \tau$,

$$w_u(j) \le M_\delta \xi^j, \quad i \in N_0,\ 0 \le j \le n. \tag{4.49}$$

Then we can obtain

$$\begin{aligned} |u_{i,n+1}| &\le |a_{i,n}||u_{i+1,n}| + |b_{i,n}||u_{i,n}| + |p_{i,n}||u_{i-\sigma,n-\tau}| \\ &\le (|\hat{a}_{i,n}| + |\hat{b}_{i,n}| + \xi^{-\tau+1}|p_{i,n}|)M_\delta \xi^{n-1} \le M_\delta \xi^{n+1}. \end{aligned} \tag{4.50}$$

By induction, we have $w_u(n) \le M_\delta \xi^n$ for $n \ge 0$. The proof is completed. □

Now we consider the case $\sigma, \tau > 0$.

Let

$$
\begin{aligned}
D_1 &= \{(i,j) \mid 0 \le i < \sigma,\ 0 \le j \le \tau\},\\
D_2 &= \{(i,j) \mid 0 \le i < \sigma,\ j > \tau\},\\
D_3 &= \{(i,j) \mid i \ge \sigma,\ 0 \le j \le \tau\},\\
D_4 &= \{(i,j) \mid i \ge \sigma,\ j > \tau\}.
\end{aligned}
\tag{4.51}
$$

Obviously, D_1 is a finite set, D_2, D_3, and D_4 are infinite sets, D_1, D_2, D_3, and D_4 are disjoint, and

$$
N_0^2 = N_0 \times N_0 = D_1 + D_2 + D_3 + D_4,
\tag{4.52}
$$

where $A + B$ denotes the union of any two subsets A and B of Z^2.

Theorem 4.12. *Assume that there exists a constant* $\xi \in (0,1)$ *such that*

$$
\begin{aligned}
|a_{i,j}| + |b_{i,j}| + \xi^{-j}|p_{i,j}| &\le \xi, \quad (i,j) \in D_2,\\
|a_{i,j}| + |b_{i,j}| + \xi^{-\tau}|p_{i,j}| &\le \xi, \quad (i,j) \in D_3 + D_4,
\end{aligned}
\tag{4.53}
$$

then (4.14) *is exponentially asymptotically stable.*

Proof. For a given solution $\{u_{i,j}\}$ of (4.14), it is obvious that there exists a constant $\theta > 1$ such that

$$
|a_{i,j}| + |b_{i,j}| + \xi^{-\tau}|p_{i,j}| \le \theta, \quad (i,j) \in D_1.
\tag{4.54}
$$

Let

$$
B_1 = \max_{(i,j)\in D_1} \{u_{i,j}\},
\tag{4.55}
$$

then B_1 is a finite constant. For the given $\xi \in (0,1)$, any $\delta > 0$ and $\varphi \in S_\delta$, it is easy to see that there exists a positive constant $M_1 > B_1\xi^{-\tau-1} > 0$ such that

$$
|u_{i,j}| \le B_1 \le M_1\xi^{j+1}, \quad (i,j) \in D_1.
\tag{4.56}
$$

For $j = 0$ and any $i \ge \sigma$, then $(i,j) \in D_3$, and there exists a positive constant

$$
M_2 \ge \max\{\xi^{-1}\|\varphi\|, \theta\|\varphi\|, M_1\}
\tag{4.57}
$$

such that

$$
\begin{aligned}
|u_{i,1}| &\le |a_{i,0}||u_{i+1,0}| + |b_{i,0}||u_{i,0}| + |p_{i,0}||u_{i-\sigma,-\tau}|\\
&\le (|a_{i,0}| + |b_{i,0}| + |p_{i,0}|)\|\varphi\| \le M_2\xi^2.
\end{aligned}
\tag{4.58}
$$

Assume that for some fixed positive integer $0 \le n \le \tau$ and any $i \ge \sigma$,

$$|u_{i,j}| \le M_2\xi^{j+1}, \quad 0 \le j \le n,\ i \ge \sigma. \tag{4.59}$$

Then $(i,n) \in D_3$,

$$\begin{aligned} |u_{i,n+1}| &\le |a_{i,n}||u_{i+1,n}| + |b_{i,n}||u_{i,n}| + |p_{i,n}||u_{i-\sigma,n-\tau}| \\ &\le (|a_{i,n}| + |b_{i,n}|)M_2\xi^{n+1} + |p_{i,n}|\|\varphi\| \le M_2\xi^{n+2}. \end{aligned} \tag{4.60}$$

Hence by induction, we have

$$|u_{i,j}| \le M_2\xi^{j+1} \quad \text{for } 0 \le j \le \tau+1,\ i \ge \sigma. \tag{4.61}$$

Let $M_\delta = \max\{\theta M_1, \theta M_2\}$, then from (4.56) and (4.61), we have

$$|u_{i,j}| \le M_\delta\xi^j, \quad (i,j) \in D_1 + D_3. \tag{4.62}$$

In view of (4.56) and (4.61), we have, for $0 \le i < \sigma$,

$$\begin{aligned} |u_{i,\tau+1}| &\le |a_{i,\tau}||u_{i+1,\tau}| + |b_{i,\tau}||u_{i,\tau}| + |p_{i,\tau}||u_{i-\sigma,0}| \\ &\le (|a_{i,\tau}| + |b_{i,\tau}|)M_2\xi^{\tau+1} + |p_{i,\tau}|\|\varphi\| \le M_\delta\xi^{\tau+1}. \end{aligned} \tag{4.63}$$

Hence

$$|u_{i,j}| \le M_\delta\xi^j, \quad 0 \le j \le \tau+1,\ i \ge 0. \tag{4.64}$$

Let

$$\begin{aligned} \overline{D}_1 &= \{(i,j) \mid i \ge 0,\ 0 \le j \le \tau\} = D_1 + D_3, \\ \overline{D}_k &= \{(i,j) \mid i \ge 0,\ (k-1)\tau < j \le k\tau\}, \quad k = 2,3,\dots, \end{aligned} \tag{4.65}$$

then from (4.62), we have

$$|u_{i,j}| \le M_\delta\xi^j, \quad (i,j) \in \overline{D}_1. \tag{4.66}$$

Assume that for some fixed positive integer $k > 0$,

$$|u_{i,j}| \le M_\delta\xi^j, \quad (i,j) \in \bigcup_{s=1}^{k}\overline{D}_s. \tag{4.67}$$

In the following, we will assert that

$$|u_{i,j}| \le M_\delta\xi^j, \quad (i,j) \in \overline{D}_{k+1}, \tag{4.68}$$

holds. For any $i \geq \sigma$, we have

$$\begin{aligned} |u_{i,k\tau+1}| &\leq |a_{i,k\tau}||u_{i+1,k\tau}| + |b_{i,k\tau}||u_{i,k\tau}| + |p_{i,k\tau}||u_{i-\sigma,(k-1)\tau}| \\ &\leq (|a_{i,k\tau}| + |b_{i,k\tau}| + \xi^{-\tau}|p_{i,k\tau}|)M_\delta \xi^{k\tau} \leq M_\delta \xi^{k\tau+1}; \end{aligned} \tag{4.69}$$

and for any $0 \leq i < \sigma$, if $k = 1$, then

$$\begin{aligned} |u_{i,k\tau+1}| &\leq |a_{i,k\tau}||u_{i+1,k\tau}| + |b_{i,k\tau}||u_{i,k\tau}| + |p_{i,k\tau}||u_{i-\sigma,(k-1)\tau}| \\ &\leq (|a_{i,k\tau}| + |b_{i,k\tau}| + \xi^{-k\tau}|p_{i,k\tau}|)M_2 \xi^{k\tau+1} \leq M_\delta \xi^{k\tau+1}; \end{aligned} \tag{4.70}$$

if $k > 1$, then

$$\begin{aligned} |u_{i,k\tau+1}| &\leq |a_{i,k\tau}||u_{i+1,k\tau}| + |b_{i,k\tau}||u_{i,k\tau}| + |p_{i,k\tau}||u_{i-\sigma,(k-1)\tau}| \\ &\leq (|a_{i,k\tau}| + |b_{i,k\tau}| + \xi^{-k\tau}|p_{i,k\tau}|)M_\delta \xi^{k\tau} \leq M_\delta \xi^{k\tau+1}. \end{aligned} \tag{4.71}$$

Hence

$$|u_{i,j}| \leq M_\delta \xi^j, \quad 0 \leq j \leq k\tau + 1,\ i \geq 0. \tag{4.72}$$

Especially, $|u_{i,k\tau+1}| \leq M_\delta \xi^{k\tau+1}$ for any $i \geq 0$.

Assume that for some fixed positive integer $k\tau < n \leq (k+1)\tau$ and any $i \geq 0$,

$$|u_{i,j}| \leq M_\delta \xi^j, \quad k\tau < j \leq n,\ i \geq 0. \tag{4.73}$$

Then for any $i \geq \sigma$, $(i,n) \in D_4$. In view of (4.67), we have

$$\begin{aligned} |u_{i,n+1}| &\leq |a_{i,n}||u_{i+1,n}| + |b_{i,n}||u_{i,n}| + |p_{i,n}||u_{i-\sigma,n-\tau}| \\ &\leq (|a_{i,n}| + |b_{i,n}| + \xi^{-\tau}|p_{i,n}|)M_\delta \xi^n \leq M_\delta \xi^{n+1}. \end{aligned} \tag{4.74}$$

Hence by induction, we have

$$|u_{i,j}| \leq M_\delta \xi^j, \quad i \geq 0,\ k\tau < j \leq (k+1)\tau. \tag{4.75}$$

Thus we have

$$|u_{i,j}| \leq M_\delta \xi^j, \quad (i,j) \in \bigcup_{s=1}^{k+1} \overline{D}_s. \tag{4.76}$$

By induction, we can see that (4.67) holds for any positive integer $k > 0$. Since

$$N_0^2 = \overline{D}_1 + \overline{D}_2 + \cdots = \bigcup_{s=1}^{\infty} \overline{D}_s, \tag{4.77}$$

then we can obtain

$$|u_{i,j}| \le M_\delta \xi^j, \quad (i,j) \in N_0^2. \tag{4.78}$$

The proof is completed. □

If the assumption of Theorem 4.12 does not hold, then we have the following result.

Theorem 4.13. *Let $\tau > 0$ and let $\xi \in (0,1)$ be a constant,*

$$\hat{A}_{i,j} = |a_{i,j}| + |b_{i,j}| + \xi^{-\tau}|p_{i,j}| \tag{4.79}$$

for any $i, j \in N_0$, and

$$\bar{a}_{i,j} = a_{i,j}\hat{A}_{i+1,j-1}, \qquad \bar{b}_{i,j} = b_{i,j}\hat{A}_{i,j-1} \tag{4.80}$$

for any $i \in N_0$ and $j > 0$. Assume that there exists a constant $C > 1$ such that

$$\begin{gathered} |a_{i,0}| + |b_{i,0}| + |p_{i,0}| \le C\xi, \quad i \in N_0, \\ |\bar{a}_{i,j}| + |\bar{b}_{i,j}| + \xi^{-j+1}|p_{i,j}| \le \xi^2, \quad (i,j) \in D_2, \\ |\bar{a}_{i,j}| + |\bar{b}_{i,j}| + \xi^{-\tau+1}|p_{i,j}| \le \xi^2, \quad (i,j) \in D_3 + D_4. \end{gathered} \tag{4.81}$$

Then (4.14) is exponentially asymptotically stable.

Proof. Similar to the proof of Theorem 4.12, for a given solution $\{u_{i,j}\}$ of (4.14), the given $\xi \in (0,1)$, any $\delta > 0$ and $\varphi \in S_\delta$, there exists a positive constant $M_1 > \xi^{-2}\|\varphi\|$ such that

$$|u_{i,j}| \le M_1 \xi^{j+2}, \quad (i,j) \in D_1. \tag{4.82}$$

It is obvious that there exists a constant $\theta > C\xi^{-2}$ such that

$$|\bar{a}_{i,j}| + |\bar{b}_{i,j}| + \xi^{-\tau+1}|p_{i,j}| \le A\xi^2, \quad 0 \le i < \sigma,\ 0 < j \le \tau, \tag{4.83}$$

where A is a constant. For $j = 0$ and any $i \ge \sigma$, then $(i,j) \in D_3$ and there exists a positive constant

$$M_2 \ge \max\{C\xi^{-2}\|\varphi\|, M_1, \xi^{-\tau-1}\|\varphi\|, \theta C\|\varphi\|\} \tag{4.84}$$

such that

$$\begin{aligned} |u_{i,1}| &\le |a_{i,0}||u_{i+1,0}| + |b_{i,0}||u_{i,0}| + |p_{i,0}||u_{i-\sigma,-\tau}| \\ &\le (|a_{i,0}| + |b_{i,0}| + |p_{i,0}|)\|\varphi\| \le M_2\xi^3. \end{aligned} \tag{4.85}$$

Therefore,

$$
\begin{aligned}
|u_{i,2}| &\le |a_{i,1}||u_{i+1,1}| + |b_{i,1}||u_{i,1}| + |p_{i,1}||u_{i-\sigma,1-\tau}| \\
&\le |a_{i,1}|(|a_{i+1,0}||u_{i+2,0}| + |b_{i+1,0}||u_{i+1,0}| + |p_{i+1,0}||u_{i+1-\sigma,-\tau}|) \\
&\quad + |b_{i,1}|(|a_{i,0}||u_{i+1,0}| + |b_{i,0}||u_{i,0}| + |p_{i,0}||u_{i-\sigma,-\tau}|) \\
&\quad + |p_{i,1}||u_{i-\sigma,1-\tau}| \\
&\le (|\bar{a}_{i,1}| + |\bar{b}_{i,1}| + |p_{i,1}|)\|\varphi\| \le M_2\xi^4.
\end{aligned} \tag{4.86}
$$

Assume that for some fixed positive integer $1 < n \le \tau$ and any $i \ge \sigma$,

$$
|u_{i,j}| \le M_2\xi^{j+2}, \quad 1 < j \le n,\ i \ge \sigma. \tag{4.87}
$$

Then $(i, n) \in D_3$ and

$$
\begin{aligned}
|u_{i,n+1}| &\le |a_{i,n}||u_{i+1,n}| + |b_{i,n}||u_{i,n}| + |p_{i,n}||u_{i-\sigma,n-\tau}| \\
&\le |a_{i,n}|(|a_{i+1,n-1}||u_{i+2,n-1}| + |b_{i+1,n-1}||u_{i+1,n-1}| \\
&\qquad + |p_{i+1,n-1}||u_{i+1-\sigma,n-1-\tau}|) \\
&\quad + |b_{i,n}|(|a_{i,n-1}||u_{i+1,n-1}| + |b_{i,n-1}||u_{i,n-1}| + |p_{i,n-1}||u_{i-\sigma,n-1-\tau}|) \\
&\quad + |p_{i,n}||u_{i-\sigma,n-\tau}|.
\end{aligned} \tag{4.88}
$$

Thus from (4.82) and (4.88), we have

$$
\begin{aligned}
|u_{i,n+1}| &\le |a_{i,n}|(|a_{i+1,n-1}| + |b_{i+1,n-1}| + \xi^{-\tau}|p_{i+1,n-1}|)M_2\xi^{n+1} \\
&\qquad + |b_{i,n}|(|a_{i,n-1}| + |b_{i,n-1}| + \xi^{-\tau}|p_{i,n-1}|)M_2\xi^{n+1} + |p_{i,n}|\|\varphi\| \\
&\le (|\bar{a}_{i,n}| + |\bar{b}_{i,n}| + \xi^{-n+1}|p_{i,n}|)M_2\xi^{n+1} \le M_2\xi^{n+3}.
\end{aligned} \tag{4.89}
$$

Hence, by induction, we have

$$
|u_{i,j}| \le M_2\xi^{j+2}, \quad 0 \le j \le \tau + 1,\ i \ge \sigma. \tag{4.90}
$$

Let $M_\delta = \max\{\theta M_1, \theta M_2\}$, then from (4.82) and (4.90), we have

$$
|u_{i,j}| \le M_\delta\xi^j, \quad (i, j) \in D_1 + D_3. \tag{4.91}
$$

In view of (4.82)–(4.91), we have for $0 \le i < \sigma$,

$$
|u_{i,\tau+1}| \le (|\bar{a}_{i,\tau}| + |\bar{b}_{i,\tau}|)M_2\xi^{\tau+1} + |p_{i,\tau}|\|\varphi\| \le M_\delta\xi^{\tau+3} \le M_\delta\xi^{\tau+1}. \tag{4.92}
$$

Hence in view of (4.90),

$$|u_{i,j}| \le M_\delta \xi^j, \quad 0 \le j \le \tau + 1,\ i \ge 0. \tag{4.93}$$

Let the subsets $\{\overline{D}_k\}$ of Z^2 be defined by (4.65), then from (4.93), we have

$$|u_{i,j}| \le M_\delta \xi^j, \quad (i,j) \in \overline{D}_1. \tag{4.94}$$

Assume that for some fixed positive integer $k > 0$,

$$|u_{i,j}| \le M_\delta \xi^j, \quad (i,j) \in \bigcup_{s=1}^{k} \overline{D}_s. \tag{4.95}$$

Then from (4.88) and (4.95), for any $i \ge \sigma$, we have

$$|u_{i,k\tau+1}| \le \left(|\overline{a}_{i,k\tau}| + |\overline{b}_{i,k\tau}| + \xi^{-\tau+1}|p_{i,k\tau}|\right) M_\delta \xi^{k\tau-1} \le M_\delta \xi^{k\tau+1}; \tag{4.96}$$

and for $0 \le i < \sigma$, if $k = 1$, then

$$|u_{i,k\tau+1}| \le \left(|\overline{a}_{i,k\tau}| + |\overline{b}_{i,k\tau}| + \xi^{-k\tau+1}|p_{i,k\tau}|\right) M_2 \xi^{k\tau-1} \le M_\delta \xi^{k\tau+1}, \tag{4.97}$$

if $k > 1$, then

$$|u_{i,k\tau+1}| \le \left(|\overline{a}_{i,k\tau}| + |\overline{b}_{i,k\tau}| + \xi^{-k\tau+1}|p_{i,k\tau}|\right) M_\delta \xi^{k\tau-1} \le M_\delta \xi^{k\tau+1}. \tag{4.98}$$

Hence

$$|u_{i,j}| \le M_\delta \xi^j \quad \text{for } 0 \le j \le k\tau + 1 \text{ and any } i \ge 0. \tag{4.99}$$

Assume that for some fixed positive integer $k\tau < n \le (k+1)\tau$ and any $i \ge 0$,

$$|u_{i,j}| \le M_\delta \xi^j \quad \text{for } k\tau < j \le n \text{ and any } i \ge 0. \tag{4.100}$$

Then for any $i \ge \sigma$, from (4.88) and (4.95), we have

$$|u_{i,n+1}| \le \left(|\overline{a}_{i,n}| + |\overline{b}_{i,n}| + \xi^{-\tau+1}|p_{i,n}|\right) M_\delta \xi^{n-1} \le M_\delta \xi^{n+1}; \tag{4.101}$$

and for any $0 \le i < \sigma$, we have

$$|u_{i,n+1}| \le \left(|\overline{a}_{i,n}| + |\overline{b}_{i,n}| + \xi^{-n+1}|p_{i,n}|\right) M_\delta \xi^{n-1} \le M_\delta \xi^{n+1}. \tag{4.102}$$

Hence by induction, we have

$$|u_{i,j}| \leq M_\delta \xi^j, \quad i \geq 0,\ k\tau < j \leq (k+1)\tau. \tag{4.103}$$

Thus we have

$$|u_{i,j}| \leq M_\delta \xi^j, \quad (i,j) \in \bigcup_{s=1}^{k+1} \overline{D}_s. \tag{4.104}$$

Hence by induction, we can obtain

$$|u_{i,j}| \leq M_\delta \xi^j, \quad (i,j) \in N_0^2. \tag{4.105}$$

The proof is completed. □

Example 4.14. Consider the partial difference equation

$$u_{i,j+1} = a_{i,j}u_{i+1,j} + b_{i,j}u_{i,j} + p_{i,j}u_{i-1,j-1}, \tag{4.106}$$

where

$$a_{i,j} = \frac{1}{2}, \qquad b_{i,j} = \frac{1}{4}, \qquad p_{i,j} = \frac{1}{8^{j+1}}. \tag{4.107}$$

Since $\sigma = 1$ and $\tau = 1$, Theorem 4.11 cannot assert that (4.106) is exponentially asymptotically stable.

But if we let $\xi = 7/8$, then for any $(i,j) \in D_2$, we have

$$|a_{i,j}| + |b_{i,j}| + \xi^{-j}|p_{i,j}| = \frac{1}{2} + \frac{1}{4} + \frac{7^{-j}}{8^{-j}} \cdot \frac{1}{8^{j+1}} \leq \frac{43}{56} < \frac{7}{8} = \xi; \tag{4.108}$$

and for $(i,j) \in D_3 + D_4$, we have

$$|a_{i,j}| + |b_{i,j}| + \xi^{-1}|p_{i,j}| = \frac{1}{2} + \frac{1}{4} + \frac{8}{7} \cdot \frac{1}{8^{j+1}} \leq \frac{43}{56} < \frac{7}{8} = \xi. \tag{4.109}$$

Hence (4.53) holds, by Theorem 4.12, (4.106) is exponentially asymptotically stable.

Example 4.15. Consider the partial difference equation

$$u_{i,j+1} = a_{i,j}u_{i+1,j} + b_{i,j}u_{i,j} + p_{i,j}u_{i-1,j-1}, \tag{4.110}$$

where

$$a_{i,j} = \frac{1}{2} + \frac{(-1)^i}{2}, \qquad b_{i,j} = \frac{1}{8}, \qquad p_{i,j} = \frac{1}{8^{j+1}}. \tag{4.111}$$

It is obvious that

$$\frac{5}{4} \geq |a_{i,j}| + |b_{i,j}| + |p_{i,j}| \geq \frac{5}{8} + \frac{(-1)^i}{2}, \quad (i,j) \in N_0^2. \tag{4.112}$$

Hence (4.53) does not hold. Thus Theorem 4.12 is not applicable to (4.110). But it is easy to see that for $\xi = 7/8$,

$$\begin{aligned} \widehat{A}_{i,j} &= |a_{i,j}| + |b_{i,j}| + \xi^{-1}|p_{i,j}| = \frac{5}{8} + \frac{(-1)^i}{2} + \frac{1}{7\cdot 8^j}, \\ |\overline{a}_{i,j}| &= |a_{i,j}\widehat{A}_{i+1,j}| = \frac{1}{16} + \frac{(-1)^i}{16} + \frac{1}{14\cdot 8^j} + \frac{(-1)^i}{14\cdot 8^j} \leq \frac{1}{4}, \\ |\overline{b}_{i,j}| &= |b_{i,j}\widehat{A}_{i,j-1}| = \frac{5}{64} + \frac{(-1)^i}{16} + \frac{1}{7\cdot 8^j} \leq \frac{3}{16}. \end{aligned} \tag{4.113}$$

Hence for any $(i,j) \in D_2$,

$$|\overline{a}_{i,j}| + |\overline{b}_{i,j}| + \xi^{-j}|p_{i,j}| \leq \frac{9}{16} \leq \frac{7}{8} \tag{4.114}$$

for $(i,j) \in D_3 + D_4$,

$$|\overline{a}_{i,j}| + |\overline{b}_{i,j}| + \xi^{-1}|p_{i,j}| \leq \frac{9}{16} \leq \frac{7}{8}. \tag{4.115}$$

It is obvious that for any $i \geq 0$,

$$|\overline{a}_{i,0}| + |\overline{b}_{i,0}| + \xi^{-1}|p_{i,0}| \leq \frac{9}{16} \leq \frac{7}{8}. \tag{4.116}$$

Therefore, by Theorem 4.13, (4.110) is exponentially asymptotically stable.

Theorem 4.16. *Assume that there exist* $\xi, \eta \in (0,1)$ *such that*

$$\xi|a_{i,j}| + |b_{i,j}| + \xi^{-\sigma}\eta^{-\tau}|p_{i,j}| \leq \eta, \quad (i,j) \in N_0^2. \tag{4.117}$$

Then (4.14) *is strongly exponentially asymptotically stable in the meaning*

$$|u_{i,j}| \leq M\|\varphi\|\xi^i\eta^j, \quad (i,j) \in N_0^2. \tag{4.118}$$

Proof. Let $h_{i,j} = \xi^i\eta^j$. We consider the equation

$$h_{i,j+1}u_{i,j+1} = a_{i,j}h_{i+1,j}u_{i+1,j} + b_{i,j}h_{i,j}u_{i,j} + p_{i,j}h_{i-\sigma,j-\tau}u_{i-\sigma,j-\tau}, \tag{4.119}$$

which equals to

$$u_{i,j+1} = [\xi a_{i,j}u_{i+1,j} + b_{i,j}u_{i,j} + \xi^{-\sigma}\eta^{-\tau}p_{i,j}u_{i-\sigma,j-\tau}]\eta^{-1}. \tag{4.120}$$

In view of (4.53) and Theorem 4.8, (4.119) is linearly stable. Hence

$$|u_{i,j}| \leq M\|\varphi\|\xi^i\eta^j, \quad (i,j)\in N_0^2. \tag{4.121}$$

The proof is complete. □

Example 4.17. Consider the partial difference equation

$$\frac{1}{16}u_{i+1,j}+u_{i,j}+\frac{1}{4}\left(\frac{13}{8}-\frac{1}{i+j+1}\right)u_{i,j+1}-\frac{1}{64(i+j+1)}u_{i-2,j-2}=0, \tag{4.122}$$

where

$$|a_{i,j}|=\frac{1}{16}, \qquad |b_{i,j}|=\frac{1}{4}\left(\frac{13}{8}-\frac{1}{i+j+1}\right), \qquad |p_{i,j}|=\frac{1}{64(i+j+1)}. \tag{4.123}$$

Let $\xi=\eta=1/2$. Then

$$\xi|a_{i,j}|+|b_{i,j}|+\xi^{-2}\eta^{-2}|p_{i,j}|=\frac{1}{32}+\frac{13}{32}<\frac{1}{2}=\eta. \tag{4.124}$$

By Theorem 4.16, the solutions of the above equation satisfy

$$|u_{i,j}| \leq M\|\varphi\|2^{-(i+j)}, \quad (i,j)\in N_0^2. \tag{4.125}$$

Now, we consider the instability of (4.14).

Let $B_{i,j}=|a_{i,j}|+|b_{i,j}|$ for any $i,j\in N_0$, and

$$\widetilde{a}_{i,j+1}=a_{i,j+1}B_{i+1,j}, \qquad \widetilde{b}_{i,j+1}=b_{i,j+1}B_{i,j}. \tag{4.126}$$

Theorem 4.18. *Assume that for some constant $r>1$, one of the following conditions holds.*

(i) $a_{i,j}\geq 0$, $b_{i,j}\geq 0$, $p_{i,j}\geq 0$ *for* $i,j\in N_0$, *and*

$$a_{i,0}+b_{i,0}\geq r, \quad \widetilde{a}_{i,j}+\widetilde{b}_{i,j}\geq r^2, \quad i\in N_0,\ j>0. \tag{4.127}$$

(ii) $a_{i,j}\leq 0$, $b_{i,j}\geq 0$, $p_{i,j}\geq 0$, $i,j\in N_0$, σ *is even and*

$$-a_{i,0}+b_{i,0}\geq r, \quad -\widetilde{a}_{i,j}+\widetilde{b}_{i,j}\geq r^2, \quad i\in N_0,\ j>0. \tag{4.128}$$

(iii) $a_{i,j}\leq 0$, $b_{i,j}\geq 0$, $p_{i,j}\leq 0$, σ *is odd and* (4.128) *holds.*

(iv) $a_{i,j}\geq 0$, $b_{i,j}\leq 0$, $p_{i,j}\geq 0$, $\sigma+\tau$ *is odd and*

$$a_{i,0}-b_{i,0}\geq r, \quad \widetilde{a}_{i,j}-\widetilde{b}_{i,j}\geq r^2, \quad i\in N_0,\ j>0. \tag{4.129}$$

(v) $a_{i,j} \geq 0,\ b_{i,j} \leq 0,\ p_{i,j} \leq 0$, *$\sigma + \tau$ is even and* (4.129) *holds.*
(vi) $a_{i,j} \leq 0,\ b_{i,j} \leq 0,\ p_{i,j} \geq 0$, *$\tau$ is odd and*

$$-a_{i,0} - b_{i,0} \geq r, \quad -\widetilde{a}_{i,j} - \widetilde{b}_{i,j} \geq r^2, \quad i \in N_0,\ j > 0. \tag{4.130}$$

(vii) $a_{i,j} \leq 0,\ b_{i,j} \leq 0,\ p_{i,j} \leq 0$, *$\tau$ is even and* (4.130) *holds.*
Then (4.14) *is unstable.*

Proof. In the following, we only give the proof for cases (i), (ii), (iv), and (vi). The other cases can be proved by the same method.

If (i) holds, we take $\varphi_{i,j} = \delta > 0$ for $(i,j) \in \Omega$; from (4.14), we have

$$u_{i,1} = a_{i,0}u_{i+1,0} + b_{i,0}u_{i,0} + p_{i,0}u_{i-\sigma,-\tau} = \delta(a_{i,0} + b_{i,0} + p_{i,0}) \geq \delta r > 0. \tag{4.131}$$

Hence from (4.14), we can obtain

$$\begin{aligned} u_{i,2} &= a_{i,1}u_{i+1,1} + b_{i,1}u_{i,1} + p_{i,1}u_{i-\sigma,1-\tau} \\ &= a_{i,1}(a_{i+1,0}u_{i+2,0} + b_{i+1,0}u_{i+1,0} + p_{i+1,0}u_{i+1-\sigma,-\tau}) \\ &\quad + b_{i,1}(a_{i,0}u_{i+1,0} + b_{i,0}u_{i,0} + p_{i,0}u_{i-\sigma,-\tau}) + p_{i,1}u_{i-\sigma,1-\tau} \\ &\geq \delta(\widetilde{a}_{i,1} + \widetilde{b}_{i,1}) \geq \delta r^2 > 0. \end{aligned} \tag{4.132}$$

Assume that for some fixed integer $n > 0$,

$$u_{i,j} \geq \delta r^j > 0, \quad i \in N_0,\ 0 < j \leq n. \tag{4.133}$$

Then from (4.14), for any $i \in N_0$, we have

$$u_{i,n+1} = a_{i,n}u_{i+1,n} + b_{i,n}u_{i,n} + p_{i,n}u_{i-\sigma,n-\tau} \geq \delta r^{n-1}(\widetilde{a}_{i,n} + \widetilde{b}_{i,n}) \geq \delta r^{n+1}. \tag{4.134}$$

By induction, we have

$$u_{i,j} \geq \delta r^j, \quad i,j \in N_0. \tag{4.135}$$

Obviously, $u_{i,j} \to \infty$ as $j \to \infty$ for any $\delta > 0$, then (4.14) is unstable.

If (ii) holds, we take $\varphi_{i,j} = (-1)^i\delta$ for $(i,j) \in \Omega$. From (4.14), we have

$$u_{i,1} = a_{i,0}u_{i+1,0} + b_{i,0}u_{i,0} + p_{i,0}u_{i-\sigma,-\tau} = (-1)^i\delta(-a_{i,0} + b_{i,0} + p_{i,0}), \quad i \in N_0. \tag{4.136}$$

Hence $(-1)^i u_{i,1} > 0$ for $i \in N_0$, and

$$|u_{i,1}| = \delta(-a_{i,0} + b_{i,0} + p_{i,0}) \geq \delta r, \quad i \in N_0. \tag{4.137}$$

Assume that for some fixed integer $n > 0$,

$$\begin{aligned} (-1)^i u_{i,j} > 0, &\quad i \geq -\sigma,\ -\tau \leq j \leq n, \\ |u_{i,j}| \geq \delta r^j, &\quad i \geq 0,\ 0 \leq j \leq n. \end{aligned} \tag{4.138}$$

Then from (4.14), we obtain

$$\begin{aligned} u_{i,n+1} &= a_{i,n}u_{i+1,n} + b_{i,n}u_{i,n} + p_{i,n}u_{i-\sigma,n-\tau} \\ &= a_{i,n}\big(a_{i+1,n-1}u_{i+2,n-1} + b_{i+1,n-1}u_{i+1,n-1} + p_{i+1,n-1}u_{i+1-\sigma,n-1-\tau}\big) \\ &\quad + b_{i,n}\big(a_{i,n-1}u_{i+1,n-1} + b_{i,n-1}u_{i,n-1} + p_{i,n-1}u_{i-\sigma,n-1-\tau}\big) + p_{i,n}u_{i-\sigma,n-\tau}. \end{aligned} \tag{4.139}$$

Hence $(-1)^i u_{i,n+1} > 0$ for $i \geq 0$, and

$$|u_{i,n+1}| \geq \delta r^{n-1}\big(-\tilde{a}_{i,n} + \tilde{b}_{i,n}\big) \geq \delta r^{n+1}, \quad i \geq 0. \tag{4.140}$$

By induction, we have

$$|u_{i,j}| \geq \delta r^j, \quad i, j \in N_0. \tag{4.141}$$

Then (4.14) is unstable.

If (iv) holds, we take $\varphi_{i,j} = (-1)^{i+j}\delta$ for $(i,j) \in \Omega$. From (4.14), we have

$$u_{i,1} = a_{i,0}u_{i+1,0} + b_{i,0}u_{i,0} + p_{i,0}u_{i-\sigma,-\tau} = (-1)^{i+1}\delta\big(a_{i,0} - b_{i,0} + p_{i,0}\big), \quad i \in N_0. \tag{4.142}$$

Hence $(-1)^{i+1}u_{i,1} > 0$ for $i \in N_0$, and

$$|u_{i,1}| = \delta\big(a_{i,0} - b_{i,0} + p_{i,0}\big) \geq \delta r, \quad i \in N_0. \tag{4.143}$$

Assume that for some fixed integer $n > 0$,

$$\begin{aligned} (-1)^{i+j} u_{i,j} > 0, &\quad i \geq -\sigma,\ -\tau \leq j \leq n, \\ |u_{i,j}| \geq \delta r^j, &\quad i \geq 0,\ 0 \leq j \leq n. \end{aligned} \tag{4.144}$$

Then from (4.14), we obtain (4.139). Hence $(-1)^{i+n+1}u_{i,n+1} > 0$ for $i \in N_0$ and

$$|u_{i,n+1}| \geq \delta r^{n-1}\big(\tilde{a}_{i,n} - \tilde{b}_{i,n}\big) \geq \delta r^{n+1}, \quad i \geq 0. \tag{4.145}$$

By induction, we have

$$|u_{i,j}| \geq \delta r^j, \quad i, j \in N_0. \tag{4.146}$$

Then (4.14) is unstable.

If (vi) holds, we take $\varphi_{i,j} = (-1)^j\delta$ for $(i,j) \in \Omega$. From (4.14), we have

$$u_{i,1} = a_{i,0}u_{i+1,0} + b_{i,0}u_{i,0} + p_{i,0}u_{i-\sigma,-\tau} = -\delta(-a_{i,0} - b_{i,0} + p_{i,0}), \quad i \in N_0. \tag{4.147}$$

Hence $-u_{i,1} > 0$ for $i \in N_0$, and

$$|u_{i,1}| = \delta(-a_{i,0} - b_{i,0} + p_{i,0}) \geq \delta r, \quad i \in N_0. \tag{4.148}$$

Assume that for some fixed integer $n > 0$,

$$\begin{gathered} (-1)^j u_{i,j} > 0, \quad i \geq -\sigma,\ -\tau \leq j \leq n, \\ |u_{i,j}| \geq \delta r^j, \quad i \geq 0,\ 0 \leq j \leq n. \end{gathered} \tag{4.149}$$

Then from (4.14), we obtain (4.139). Hence $(-1)^{n+1}u_{i,n+1} > 0$ for $i \in N_0$ and

$$|u_{i,n+1}| \geq \delta r^{n-1}(-\widetilde{a}_{i,n} - \widetilde{b}_{i,n}) \geq \delta r^{n+1}, \quad i \geq 0. \tag{4.150}$$

By induction, we have

$$|u_{i,j}| \geq \delta r^j, \quad i,j \in N_0. \tag{4.151}$$

Then (4.14) is unstable. The proof is completed. □

Remark 4.19. We compare conditions of Theorem 4.8 for the stability and conditions in Theorem 4.13 for the instability to find that there is a gap between them. How do we fill this gap? That is an open problem.

Similarly, we can prove the following result. Let

$$\begin{gathered} \hat{B}_{i,j} = |a_{i,j}| + |b_{i,j}| + r^{-\tau}|p_{i,j}|, \quad i,j \in N_0, \\ \overline{\overline{a}}_{i,j+1} = a_{i,j+1}\hat{B}_{i+1,j}, \qquad \overline{\overline{b}}_{i,j+1} = b_{i,j+1}\hat{B}_{i,j}. \end{gathered} \tag{4.152}$$

Theorem 4.20. *Assume that $\sigma = 0$ and $\tau > 0$. Let for some constant $r > 1$, one of the following conditions holds.*

(i) $a_{i,j} \geq 0,\ b_{i,j} \geq 0,\ p_{i,j} \geq 0$ *for $i,j \in N_0$, and*

$$\begin{gathered} |a_{i,0}| + |b_{i,0}| + |p_{i,0}| \geq r, \\ |\overline{\overline{a}}_{i,j}| + |\overline{\overline{b}}_{i,j}| + r^{-\tau+1}|p_{i,j}| \geq r^2, \quad i \in N_0,\ j > 0. \end{gathered} \tag{4.153}$$

(ii) $a_{i,j} \leq 0,\ b_{i,j} \geq 0,\ p_{i,j} \geq 0$ *for $i,j \in N_0$, σ is even and* (4.153) *holds.*
(iii) $a_{i,j} \leq 0,\ b_{i,j} \geq 0,\ p_{i,j} \leq 0$, *$\sigma$ is odd and* (4.153) *holds.*
(iv) $a_{i,j} \geq 0,\ b_{i,j} \leq 0,\ p_{i,j} \geq 0$, *$\sigma + \tau$ is odd and* (4.153) *holds.*
(v) $a_{i,j} \geq 0,\ b_{i,j} \leq 0,\ p_{i,j} \leq 0$, *$\sigma + \tau$ is even and* (4.153) *holds.*

(vi) $a_{i,j} \le 0,\ b_{i,j} \le 0,\ p_{i,j} \ge 0$, τ *is odd and* (4.153) *holds.*
(vii) $a_{i,j} \le 0,\ b_{i,j} \le 0,\ p_{i,j} \le 0$, τ *is even and* (4.153) *holds.*
Then (4.14) *is unstable.*

Example 4.21. Consider the partial difference equation

$$u_{i,j+1} = a_{i,j}u_{i+1,j} + b_{i,j}u_{i,j} + p_{i,j}u_{i-2,j-1}, \tag{4.154}$$

where

$$a_{i,j} = -\frac{3}{4} + \frac{(-1)^i}{2}, \qquad b_{i,j} = \frac{3}{4}, \qquad p_{i,j} = \frac{1}{8}. \tag{4.155}$$

It is easy to see that

$$B_{i,j} = |a_{i,j}| + |b_{i,j}| = \frac{3}{2} - \frac{(-1)^i}{2}. \tag{4.156}$$

From (4.155) and (4.156),

$$\begin{aligned} \widetilde{a}_{i,j} &= a_{i,j+1}B_{i+1,j} = -\frac{7}{8} + (-1)^i\frac{3}{8}, \\ \widetilde{b}_{i,j} &= b_{i,j+1}B_{i,j} = \frac{9}{8} - (-1)^i\frac{3}{8}, \end{aligned} \tag{4.157}$$

and hence

$$-\widetilde{a}_{i,j} + \widetilde{b}_{i,j} = 2 + (-1)^i\frac{3}{4} \ge \frac{5}{4} > 1. \tag{4.158}$$

Thus condition (ii) of Theorem 4.18 holds. By Theorem 4.18, (4.154) is unstable.

4.2.2. Stability of linear PDEs with continuous arguments

Consider the partial difference equation with continuous arguments of the form

$$u(x, y+t) = a(x, y)u(x+s, y) + b(x, y)u(x, y) + p(x, y)u(x-\sigma, y-\tau), \tag{4.159}$$

where $s > 0$, $t > 0$, σ and τ are nonnegative constants, $a(x, y)$, $b(x, y)$, and $p(x, y)$ are real functions defined on $x \ge 0$ and $y \ge 0$.

By a solution of (4.159) we mean a real function $u(x, y)$ which is defined for $x \ge -\sigma$ and $y \ge -\tau$, and satisfies (4.159) for $x \ge 0$ and $y \ge 0$.

Let h be a real number, $R = (-\infty, \infty)$, $R_h = [h, +\infty)$, and $\Omega = R_{-\sigma} \times R_{-\tau} \setminus R_0 \times R_0$. It is easy to construct by iterative method a function $u(x, y)$ which equals $\varphi(x, y)$ on Ω and satisfies (4.159) on $R_0 \times R_0$. Obviously, the solution of the initial value problem of (4.159) is unique.

For any initial function $\varphi(x, y)$ on Ω, let

$$\|\varphi\| = \sup_{(x,y)\in\Omega} |\varphi(x, y)|. \tag{4.160}$$

For any positive real number $H > 0$, let $S_H = \{\varphi \mid \|\varphi\| < H\}$.

Stability and exponential asymptotic stability are defined as follows.

Definition 4.22. Equation (4.159) is said to be linearly stable if there exists a constant $M > 0$ such that every solution of (4.159) satisfies

$$|u(x, y)| \le M\|\varphi\|, \quad x, y \in R_0. \tag{4.161}$$

Equation (4.159) is said to be stable, if for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varphi \in S_\delta$ implies that the corresponding solution $u(x, y)$ satisfies

$$|u(x, y)| < \varepsilon, \quad x, y \in R_0. \tag{4.162}$$

From the above definition, it is obvious that (4.159) is linearly stable which implies that it is stable.

Definition 4.23. Equation (4.159) is said to be exponentially asymptotically stable if, for any $\delta > 0$, there exist a positive constant M_δ and a real number $\xi \in (0, 1)$ such that $\varphi \in S_\delta$ implies that

$$|u(x, y)| \le M_\delta \xi^y, \quad x, y \in R_0, \tag{4.163}$$

where $u(x, y)$ is a solution of (4.159) with the initial function $\varphi(x, y)$.

Let $V(u, x, y) : R \times R_0^2 \to R^+ = [0, \infty)$. If for any solution $u(x, y)$ of (4.159), there exists a constant $c > 0$ such that

$$V(u, x, y) \ge c|u(x, y)|, \quad (x, y) \in R_0^2, \tag{4.164}$$

then $V(u, x, y)$ is said to be a positive Liapunov function.

The following result holds obviously.

Lemma 4.24. *If for any solution $u(x, y)$ of* (4.159) *there exist a positive Liapunov function $V(u, x, y)$ and a constant $M > 0$ such that*

$$V(u, x, y) \le M\|\varphi\|, \quad (x, y) \in R_0^2, \tag{4.165}$$

where $u(x, y)$ is a solution of (4.159) *with the initial function $\varphi(x, y)$, then* (4.159) *is linearly stable.*

Theorem 4.25. *Assume that*

$$|a(x,y)| + |b(x,y)| + |p(x,y)| \le 1 \quad \forall x, y \in R_0. \tag{4.166}$$

Then (4.159) *is linearly stable.*

Proof. For a given solution $u(x,y)$ of (4.159), let

$$V(u,x,y) = \max_{x\ge 0} |u(x,y)| \quad \text{for } y \ge 0, \qquad w_u(y) = V(u,x,y). \tag{4.167}$$

Obviously, for any $x \ge 0$ and $t \le y < 2t$, we have $(x+s, y-t)$, $(x, y-t)$, $(x-\sigma, y-t-\tau) \in \Omega$. Thus from (4.159), for any $x \in R_0$ and $y \in [t, 2t)$, we obtain

$$\begin{aligned} |u(x,y)| &\le |a(x,y-t)|\,|u(x+s,y-t)| + |b(x,y-t)|\,|u(x,y-t)| \\ &\quad + |p(x,y-t)|\,|u(x-\sigma,y-t-\tau)| \\ &\le \big(|a(x,y-t)| + |b(x,y-t)| + |p(x,y-t)|\big)\cdot\|\varphi\| \le \|\varphi\|. \end{aligned} \tag{4.168}$$

Hence $|u(x,y)| \le w_u(y) \le \|\varphi\|$ for any $x \in R_0$ and $y \in [0, 2t)$. Assume that for some fixed integer $n > 1$,

$$w_u(y) \le \|\varphi\| \quad \text{for any } y \in [0, nt). \tag{4.169}$$

Then for any $y \in [nt, (n+1)t)$, we can obtain $y-t \in [0, nt)$ and $y-t-\tau \in [-\tau, nt)$, and then

$$\begin{aligned} |u(x,y)| &\le |a(x,y-t)|\,|u(x+s,y-t)| + |b(x,y-t)|\,|u(x,y-t)| \\ &\quad + |p(x,y-t)|\,|u(x-\sigma,y-t-\tau)| \\ &\le \big(|a(x,y-t)| + |b(x,y-t)| + |p(x,y-t)|\big) \\ &\quad \times \max\big\{|u(x+s,y-t)|, |u(x,y-t)|, |u(x-\sigma,y-t-\tau)|\big\} \\ &\le \|\varphi\|. \end{aligned} \tag{4.170}$$

By induction, $w_u(y) \le \|\varphi\|$ for any $y > 0$. Hence by Lemma 4.24, (4.159) is linearly stable. The proof is complete. □

Example 4.26. Consider the partial difference equation

$$u(x,y+3) = a(x,y)u(x+2,y) + b(x,y)u(x,y) + p(x,y)u(x-1,y-1), \tag{4.171}$$

where

$$a(x,y) = -\frac{x+y}{3(x+y+1)}, \quad b(x,y) = \frac{x^2+y^2}{3(x^2+y^2+1)}, \quad p(x,y) = \frac{1}{3} \quad \text{for } (x,y) \in R_0^2. \tag{4.172}$$

It is easy to see that $|a(x,y)| + |b(x,y)| + |p(x,y)| \le 1$ for any $(x,y) \in R_0^2$. Hence by Theorem 4.25, (4.171) is linearly stable.

If (4.166) does not hold, then we can obtain the following three results.

Theorem 4.27. *For any $y \ge 0$, let*

$$\overline{d}(y) = \max_{x \ge 0} \{ |a(x,y)| + |b(x,y)| + |p(x,y)| \}, \qquad d(y) = \max(1, \overline{d}(y)), \tag{4.173}$$

and $d(y) = 1 + r(y)$ for any $y \in R_0$. If there exists a positive number $M > 0$ such that for any $y \in [0,t)$,

$$\sum_{i=0}^{\infty} r(y+it) < M, \tag{4.174}$$

then (4.159) *is linearly stable.*

Proof. Similar to the proof of Theorem 4.25, by induction we can obtain

$$w_u(y) \le \left(\prod_{k=0}^{[y/t]} d(y-kt) \right) \|\varphi\| \quad \text{for any } y \ge 0. \tag{4.175}$$

Hence,

$$\begin{aligned} \ln w_u(y) &\le \ln \|\varphi\| + \sum_{k=0}^{[y/t]} \ln d(y-kt) \\ &= \ln \|\varphi\| + \sum_{k=0}^{[y/t]} \ln (1 + r(y-kt)) \\ &\le \ln \|\varphi\| + \sum_{k=0}^{\infty} r(\overline{y}+kt) \\ &\le \ln \|\varphi\| + M, \end{aligned} \tag{4.176}$$

where $\overline{y}$ is a certain constant in the interval $[0,t)$. Hence,

$$w_u(y) \le \|\varphi\| \exp(M) = \overline{M}\|\varphi\| \quad \text{for any } y \ge 0, \tag{4.177}$$

where $\overline{M} = \exp(M)$. The proof is complete. □

Let $A(x,y) = |a(x,y)| + |b(x,y)| + |p(x,y)|$ for any $(x,y) \in R_0^2$, and

$$\overline{a}(x,y+t) = a(x,y+t)A(x+s,y), \qquad \overline{b}(x,y+t) = b(x,y+t)A(x,y). \tag{4.178}$$

Theorem 4.28. *Assume that there exists a constant $C > 1$ such that*

$$\begin{gathered} |a(x,y)| + |b(x,y)| + |p(x,y)| \le C \quad \textit{for any } x \in R_0,\ y \in [0,t), \\ |\overline{a}(x,y)| + |\overline{b}(x,y)| + |p(x,y)| \le 1 \quad \textit{for } x \in R_0,\ y \in [t,\infty). \end{gathered} \tag{4.179}$$

Then (4.159) *is linearly stable.*

Proof. For a given solution $u(x,y)$ of (4.159), let $V(u,x,y)$ and $w_u(y)$ be defined in (4.167). From (4.159), for any $x \in R_0$ and $y \in [t,2t)$, we have $y-t \in [0,t)$ and

$$\begin{aligned} u(x,y) \le{} & \big(|a(x,y-t)| + |b(x,y-t)| + |p(x,y-t)|\big) \\ & \times \max\big\{|u(x+s,y-t)|, |u(x,y-t)|, |u(x-\sigma,y-t-\tau)|\big\} \le C\|\varphi\|. \end{aligned} \tag{4.180}$$

Hence $w_u(y) \le C\|\varphi\|$ for $y \in [0,2t)$. From (4.159), for any $y \ge 2t$, we have

$$\begin{aligned} |u(x,y)| \le{} & |a(x,y-t)||u(x+s,y-t)| + |b(x,y-t)||u(x,y-t)| \\ & + |p(x,y-t)||u(x-\sigma,y-t-\tau)| \\ \le{} & |a(x,y-t)|\big(|a(x+s,y-2t)||u(x+2s,y-2t)| \\ & \qquad + |b(x+s,y-2t)||u(x+s,y-2t)| \\ & \qquad + |p(x+s,y-2t)||u(x+s-\sigma,y-2t-\tau)|\big) \\ & + |b(x,y-t)|\big(|a(x,y-2t)||u(x+s,y-2t)| \\ & \qquad + |b(x,y-2t)||u(x,y-2t)| \\ & \qquad + |p(x,y-2t)||u(x-\sigma,y-2t-\tau)|\big) \\ & + |p(x,y-t)||u(x-\sigma,y-t-\tau)|. \end{aligned} \tag{4.181}$$

Hence from (4.181), for any $x \in R_0$ and $y \in [2t, 3t)$,

$$|u(x,y)| \leq \left(|\overline{a}(x,y-t)| + |\overline{b}(x,y-t)| + |p(x,y-t)|\right) \cdot C\|\varphi\| \leq C\|\varphi\|. \tag{4.182}$$

Thus $w_u(y) \leq C\|\varphi\|$ for $y \in [0, 3t)$. Assume that for some fixed integer $n > 2$,

$$w_u(y) \leq C\|\varphi\|, \quad y \in [0, nt). \tag{4.183}$$

Then in view of (4.181), for any $x \in R_0$ and $y \in [nt, (n+1)t)$, we can obtain

$$|u(x,y)| \leq \left(|\overline{a}(x,y-t)| + |\overline{b}(x,y-t)| + |p(x,y-t)|\right) \cdot C\|\varphi\| \leq C\|\varphi\|. \tag{4.184}$$

By induction, $w_u(y) \leq C\|\varphi\|$ for any $y \geq 0$. Hence $|u(x,y)| \leq w_u(y) \leq C\|\varphi\|$ for all $(x,y) \in R_0^2$, that is, (4.159) is linearly stable. The proof is complete. □

Similar to the proof of Theorems 4.27-4.28, we can obtain the following result.

Theorem 4.29. *Let*

$$\begin{aligned} \overline{d}(y) &= \max_{x \geq 0}\left\{|a(x,y)| + |b(x,y)| + |p(x,y)|\right\} \quad \text{for } y \in [0,t), \\ \overline{d}(y) &= \max_{x \geq 0}\left\{|\overline{a}(x,y)| + |\overline{b}(x,y)| + |p(x,y)|\right\} \quad \text{for } y \in [t,\infty), \end{aligned} \tag{4.185}$$

and $d(y) = \max(1, \overline{d}(y)) = 1 + r(y)$ *for* $y \geq 0$. *If there exists a positive constant* $M > 0$ *such that for any* $y \in [0,t)$,

$$\sum_{i=0}^{\infty} r(y + it) < M, \tag{4.186}$$

then (4.159) *is linearly stable.*

Example 4.30. Consider the partial difference equation

$$u(x, y+2) = a(x,y)u(x+3,y) + b(x,y)u(x,y) + p(x,y)u(x-1,y-1), \tag{4.187}$$

where

$$a(x,y) = \frac{1}{3} + \frac{1}{y^2+1}, \quad b(x,y) = -\frac{1}{3}, \quad p(x,y) = \frac{1}{3} \quad \text{for any } (x,y) \in R_0^2. \tag{4.188}$$

It is easy to see that $|a(x,y)| + |b(x,y)| + |p(x,y)| > 1$ for any $(x,y) \in R_0^2$. Hence by Theorem 4.25 it is impossible to assert that (4.187) is stable. But it is obvious that

$$d(y) = \max_{x\geq 0}\{|a(x,y)| + |b(x,y)| + |p(x,y)|\} = 1 + (y^2+1)^{-1},$$
$$r(y) = (y^2+1)^{-1}, \tag{4.189}$$

and for any $y \in [0,2)$,

$$r(y) + r(y+2) + \cdots + r(y+2n) + \cdots < \sum_{n=1}^{\infty}\frac{1}{n^2} \leq 3. \tag{4.190}$$

Hence by Theorem 4.27, (4.187) is linearly stable.

Example 4.31. Consider the partial difference equation

$$u(x,y+2) = a(x,y)u\left(x+\frac{\pi}{2},y\right) + b(x,y)u(x,y) + p(x,y)u(x-2,y-1), \tag{4.191}$$

where

$$a(x,y) = \sin x, \quad b(x,y) = \frac{1}{10}, \quad p(x,y) = \frac{1}{10} \quad \text{for } (x,y) \in R_0^2. \tag{4.192}$$

It is easy to see that $|a(x,y)| + |b(x,y)| + |p(x,y)| = |\sin x| + 0.2$ for any $x, y \in R_0$, then by Theorems 4.25 and 4.27, it is difficult to assert that (4.191) is stable. But it is easy to obtain

$$A(x,y) = |\sin x| + 0.2 \quad \text{for any } x, y \in R_0,$$
$$|a(x,y)| + |b(x,y)| + |p(x,y)| = 0.2 + |\sin x| \leq 2 = C \quad \text{for any } x \in R_0,\ y \in [0,t),$$
$$|\overline{a}(x,y+2)| = |a(x,y+2)|A\left(x+\frac{\pi}{2},y\right) = \frac{|\sin x|}{5} + \frac{|\sin 2x|}{2} \quad \text{for } x \in R_0,\ y \geq t,$$
$$|\overline{b}(x,y+2)| = |b(x,y+2)|A(x,y) = \frac{1}{50} + \frac{|\sin x|}{10} \quad \text{for } x \in R_0,\ y \in (t,+\infty). \tag{4.193}$$

Then for $x \in R_0$ and $y \geq 0$,

$$|\overline{a}(x,y+2)| + |\overline{b}(x,y+2)| + |p(x,y+2)| = \frac{3}{25} + \frac{3|\sin x|}{10} + \frac{|\sin 2x|}{2} \leq \frac{23}{25} < 1. \tag{4.194}$$

By Theorem 4.28, we can conclude that (4.191) is linearly stable.

Let $\widehat{A}(x,y) = |a(x,y)| + |b(x,y)| + \xi^{-\tau}|p(x,y)|$ for any $x, y \in R_0$ and

$$\widehat{a}(x,y+t) = a(x,y+t)\widehat{A}(x+s,y), \qquad \widehat{b}(x,y+t) = b(x,y+t)\widehat{A}(x,y). \tag{4.195}$$

Theorem 4.32. *Assume that* $\sigma = 0$, $\tau \geq 0$, *and there exist two constants* $C > 1$ *and* $\xi \in (0,1)$ *such that either*

$$|a(x,y)| + |b(x,y)| + \xi^{-\tau}|p(x,y)| \leq \xi^t \quad \forall x, y \in R_0 \tag{4.196}$$

or

$$\begin{aligned} &|a(x,y)| + |b(x,y)| + |p(x,y)| \leq C \quad \textit{for any } x \in R_0,\ y \in [0,t), \\ &|\widehat{a}(x,y)| + |\widehat{b}(x,y)| + \xi^{-(\tau+t)}|p(x,y)| \leq \xi^{2t} \quad \forall x \in R_0,\ y \geq t, \end{aligned} \tag{4.197}$$

then (4.159) *is exponentially asymptotically stable.*

Proof. For a given solution $u(x,y)$ of (4.159), let $V(u,x,y)$ and $w_u(y)$ be defined in (4.167).

If (4.196) holds, then for any $\delta > 0$ and $\varphi \in S_\delta$ there exists a constant $M_\delta \geq C\xi^{-(2t+\tau+3)}\|\varphi\| > 0$ such that for any $x \in R_0$ and $y \in [t, 2t)$,

$$\begin{aligned} |u(x,y)| &\leq |a(x,y-t)||u(x+s,y-t)| + |b(x,y-t)||u(x,y-t)| \\ &\quad + |p(x,y-t)||u(x-\sigma,y-t-\tau)| \\ &\leq (|a(x,y-t)| + |b(x,y-t)| + |p(x,y-t)|)\|\varphi\| \\ &\leq M_\delta \xi^y. \end{aligned} \tag{4.198}$$

Hence $w_u(y) \leq M_\delta \xi^y$ for any $y \in [0, 2t)$. In general, we can obtain

$$w_u(y) \leq M_\delta \xi^y \quad \text{for any } y \in [0, t+\tau). \tag{4.199}$$

Assume that for any positive integer $n \geq 0$,

$$w_u(y) \leq M_\delta \xi^y \quad \text{for any } y \in [0, t+\tau+nt). \tag{4.200}$$

Then from (4.159), for any $y \in [t+\tau+nt, t+\tau+(n+1)t)$, we have

$$|u(x,y)| \leq (|a(x,y-t)| + |b(x,y-t)| + \xi^{-\tau}|p(x,y-t)|)M_\delta \xi^{y-t} \leq M_\delta \xi^y. \tag{4.201}$$

By induction, $w_u(y) \leq M_\delta \xi^y$ for any $y \geq 0$. Hence by Lemma 4.24, (4.159) is stable.

If (4.197) holds, then for any $\delta > 0$ and $\varphi \in S_\delta$ we can obtain (4.198) for any $x \in R_0$ and $y \in [0, 2t)$, and then $w_u(y) \le M_\delta \xi^y$ for $y \in [0, 2t)$. From (4.159), for any $y \ge 2t$, we have

$$\begin{aligned}
|u(x,y)| \le\ & |a(x,y-t)|\,|u(x+s,y-t)| + |b(x,y-t)|\,|u(x,y-t)| \\
& + |p(x,y-t)|\,|u(x-\sigma,y-t-\tau)| \\
\le\ & |a(x,y-t)|\big(|a(x+s,y-2t)|\,|u(x+2s,y-2t)| \\
& \qquad + |b(x+s,y-2t)|\,|u(x+s,y-2t)| \\
& \qquad + |p(x+s,y-2t)|\,|u(x+s-\sigma,y-2t-\tau)|\big) \\
& + |b(x,y-t)|\big(|a(x,y-2t)|\,|u(x+s,y-2t)| \\
& \qquad + |b(x,y-2t)|\,|u(x,y-2t)| \\
& \qquad + |p(x,y-2t)|\,|u(x-\sigma,y-2t-\tau)|\big) \\
& + |p(x,y-t)|\,|u(x-\sigma,y-t-\tau)|.
\end{aligned} \tag{4.202}$$

Hence for any $x \in R_0$ and $y \in [2t, 3t)$,

$$|u(x,y)| \le \big(|\bar{a}(x,y-t)| + |\bar{b}(x,y-t)| + |p(x,y-t)|\big) \cdot \|\varphi\| \le M_\delta \xi^y. \tag{4.203}$$

Thus $w_u(y) \le M_\delta \xi^y$ for $y \in [0, 3t)$. In general, from (4.202), we have for any $y \in [0, 3t+\tau)$,

$$w_u(y) \le M_\delta \xi^y \quad \text{for } x \in R_0,\ y \in [0, 3t+\tau). \tag{4.204}$$

Assume that for any fixed integer $n \ge 0$,

$$w_u(y) \le M_\delta \xi^y \quad \text{for } x \in R_0,\ y \in [0, 3t+\tau+nt). \tag{4.205}$$

Then from (4.202), for any $x \in R_0$ and $y \in [3t+\tau+nt, 3t+\tau+(n+1)t)$, we can obtain

$$|u(x,y)| \le \big(|\hat{a}(x,y-t)| + |\hat{b}(x,y-t)| + \xi^{-(\tau+t)}|p(x,y-t)|\big) M_\delta \xi^{y-2t} \le M_\delta \xi^y. \tag{4.206}$$

By induction, we have $|u(x,y)| \le M_\delta \xi^y$ for $y \ge 0$. The proof is complete. □

Let $B(x,y) = |a(x,y)| + |b(x,y)|$ for any $x, y \in R_0$, and

$$\tilde{a}(x,y+t) = a(x,y+t)B(x+s,y), \qquad \tilde{b}(x,y+t) = b(x,y+t)B(x,y). \tag{4.207}$$

Theorem 4.33. *Assume that* $a(x,y) \geq 0$, $b(x,y) \geq 0$, $p(x,y) \geq 0$ *for* $x, y \in R_0$, *and there exists a positive constant* $r > 1$ *such that either*

$$a(x,y) + b(x,y) \geq r^{2t} \quad \forall x, y \in R_0 \tag{4.208}$$

or

$$\begin{aligned} a(x,y) + b(x,y) &\geq r^{2t} \quad \text{for } x \in R_0,\ y \in [0,t), \\ \widetilde{a}(x,y) + \widetilde{b}(x,y) &\geq r^{2t} \quad \text{for } x \in R_0,\ y \geq t. \end{aligned} \tag{4.209}$$

Then (4.159) *is unstable.*

Proof. If (4.208) holds, we take $\varphi(x,y) = \delta > 0$ for all $(x,y) \in \Omega$, where δ is a positive constant. In view of (4.159), for any $x \in R_0$ and $y \in [t, 2t)$, we have

$$\begin{aligned} u(x,y) &= a(x,y-t)u(x+s,y-t) + b(x,y-t)u(x,y-t) \\ &\quad + p(x,y-t)u(x-\sigma,y-t-\tau) \\ &= \delta\big(a(x,y-t) + b(x,y-t) + p(x,y-t)\big) \\ &\geq \delta \cdot r^{2t} \geq \delta \cdot r^{y} > 0. \end{aligned} \tag{4.210}$$

Assume that for some fixed integer $n > 1$,

$$u(x,y) \geq \delta \cdot r^{y} > 0 \quad \text{for } x \in R_0,\ y \in [0, nt). \tag{4.211}$$

Then from (4.159), for any $x \in R_0$ and $y \in [nt, (n+1)t)$, we have

$$\begin{aligned} u(x,y) &= a(x,y-t)u(x+s,y-t) + b(x,y-t)u(x,y-t) \\ &\quad + p(x,y-t)u(x-\sigma,y-t-\tau) \\ &\geq \delta r^{y-t}\big(a(x,y-t) + b(x,y-t)\big) \\ &\geq \delta \cdot r^{y+t} \geq \delta \cdot r^{y} > 0. \end{aligned} \tag{4.212}$$

By induction, we have

$$u(x,y) \geq \delta \cdot r^{y} \quad \text{for any } x, y \in R_0. \tag{4.213}$$

Obviously, $u(x,y) \to +\infty$ as $y \to +\infty$ for any constant $\delta > 0$, then (4.159) is unstable.

If (4.209) holds, then we also take $\varphi(x, y) = \delta > 0$ for all $(x, y) \in \Omega$. In view of (4.159), for any $x \in R_0$ and $y \in [t, 2t)$, we can also obtain (4.210). From (4.159), for any $x \in R_0$ and $y \geq 2t$, we obtain

$$\begin{aligned} u(x, y) &= a(x, y-t)u(x+s, y-t) + b(x, y-t)u(x, y-t) \\ &\quad + p(x, y-t)u(x-\sigma, y-t-\tau) \\ &= a(x, y-t)\big(a(x+s, y-2t)u(x+2s, y-2t) \\ &\qquad + b(x+s, y-2t)u(x+s, y-2t) \\ &\qquad + p(x+s, y-2t)u(x+s-\sigma, y-2t-\tau)\big) \\ &\quad + b(x, y-t)\big(a(x, y-2t)u(x+s, y-2t) \\ &\qquad + b(x, y-2t)u(x, y-2t) \\ &\qquad + p(x, y-2t)u(x-\sigma, y-2t-\tau)\big) \\ &\quad + p(x, y-t)u(x-\sigma, y-t-\tau). \end{aligned} \tag{4.214}$$

Hence from (4.209), for any $x \in R_0$ and $y \in [2t, 3t)$, we get

$$u(x, y) \geq \delta\big(\tilde{a}(x, y-t) + \tilde{b}(x, y-t)\big) \geq \delta \cdot r^{2t} \geq \delta \cdot r^{y} > 0. \tag{4.215}$$

Assume that for some fixed integer $n > 1$,

$$u(x, y) \geq \delta \cdot r^{y} > 0 \quad \text{for } x \in R_0,\ y \in [0, nt). \tag{4.216}$$

Then from (4.159) and (4.209), for any $x \in R_0$ and $y \in [nt, (n+1)t)$, we have

$$u(x, y) \geq \delta \cdot r^{y-2t}\big(\tilde{a}(x, y-t) + \tilde{b}(x, y-t)\big) \geq \delta \cdot r^{y} > 0. \tag{4.217}$$

By induction, we have

$$u(x, y) \geq \delta \cdot r^{y} \quad \text{for any } x, y \in R_0. \tag{4.218}$$

Obviously, $u(x, y) \to \infty$ as $y \to \infty$ for any constant $\delta > 0$, then (4.159) is unstable. The proof is complete. □

Similarly, we can prove the following result. Let

$$\begin{aligned} &\hat{B}(x, y) = |a(x, y)| + |b(x, y)| + r^{-\tau}|p(x, y)| \quad \text{for any } x, y \in R_0, \\ &\bar{\bar{a}}(x, y+t) = a(x, y+t)\hat{B}(x+s, y), \qquad \bar{\bar{b}}(x, y+t) = b(x, y+t)\hat{B}(x, y). \end{aligned} \tag{4.219}$$

Theorem 4.34. *Assume that* $\sigma = 0$ *and* $\tau > 0$. *If, for some constant* $r > 1$, $a(x, y) \geq 0$, $b(x, y) \geq 0$, $p(x, y) \geq 0$ *for* $x, y \in R_0$, *and*

$$\begin{aligned} |a(x,y)| + |b(x,y)| + |p(x,y)| &\geq r^{2t} \quad \text{for } x \in R_0,\ y \in [0,t), \\ |\overline{\overline{a}}(x,y)| + |\overline{\overline{b}}(x,y)| + |p(x,y)| &\geq r^{2t} \quad \text{for } x \in R_0,\ y \geq t, \end{aligned} \tag{4.220}$$

then (4.159) *is unstable.*

Example 4.35. Consider the partial difference equation

$$u(x, y+1) = a(x,y)u(x+2,y) + b(x,y)u(x,y) + p(x,y)u(x-1,y-2), \tag{4.221}$$

where

$$a(x,y) = \frac{y^2+1}{y^2+2}, \quad b(x,y) = \frac{1}{y^2+2} + \frac{1}{5}, \quad p(x,y) = \frac{1}{y^2+1} \quad \text{for } (x,y) \in R_0^2. \tag{4.222}$$

It is easy to see that $|a(x, y)| + |b(x, y)| = 1.2 = (\sqrt{1.2})^2$ for any $(x, y) \in R_0^2$. Hence by Theorem 4.33, (4.221) is unstable.

4.3. Stability of linear delay partial difference systems

Consider the system of partial difference equations

$$Z(x,y) = \begin{cases} \displaystyle\sum_{k=1}^{N} A_k(x,y) Z\big(x - p_k(x), y - q_k(y)\big), & (x,y) \in \Omega_0, \\ \varphi(x,y), & (x,y) \in \Omega_2, \end{cases} \tag{4.223}$$

where $p_k : [0,\infty) \to R_+$, $q_k : [0,\infty) \to R_+$, and $p_k(\cdot)$, $q_k(\cdot)$ are both continuous functions. $Z, \varphi \in R^n$, $A_k : \Omega_0 \to R^{n\times n}$, $k = 1, 2, \ldots, N$, are real continuous functions, and

$$\begin{aligned} \Omega_0 &= \{(x,y) \mid x \geq 0,\ y \geq 0\}, \\ \Omega_1 &= \big\{(x,y) \mid x \geq -p,\ y \geq -q\big\}, \\ \Omega_2 &= \Omega_1 \backslash \Omega_0, \end{aligned} \tag{4.224}$$

where $p > 0$, $q > 0$.

Let

$$p(x) = \max_{1\le k\le N} p_k(x),\ x \ge 0,$$
$$q(y) = \max_{1\le k\le N} q_k(y),\ y \ge 0. \tag{4.225}$$

We assume that $p(x),\ q(y)$ satisfy $p(x) \le x + p,\ q(y) \le y + q$, where $x, y \ge 0$.

For a given function $\varphi(x,y) \in R^n$ on Ω_2, it is easy to see that the initial value problem (4.223) has a unique solution $Z(x,y)$ on Ω_0.

For any $H > 0$, let

$$S_H = \{\varphi \mid \|\varphi\|_{\Omega_2} < H\}. \tag{4.226}$$

Similar to Section 4.2, we give the following definitions.

Definition 4.36. Equation (4.223) is said to be stable if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $\varphi \in S_\delta$, the corresponding solution $Z(x,y)$ of (4.223) satisfies

$$||Z(x,y)|| < \epsilon, \quad (x,y) \in \Omega_0. \tag{4.227}$$

Definition 4.37. Equation (4.223) is said to be asymptotically stable in the large if it is stable, and at the same time every solution $Z(x,y)$ with the initial function $\varphi(x,y)$, which satisfies $\sup_{(x,y)\in\Omega_2} \|\varphi(x,y)\| = c$, c is a positive constant which satisfies that $\|Z(x,y)\| \to 0$, as $\min(x,y) \to +\infty$.

Definition 4.38. Equation (4.223) is said to be exponentially asymptotically stable, if for any $\delta > 0$, there exists a real number $r \in (0,1)$ such that $\varphi \in S_\delta$ implies that

$$||Z(x,y)|| \le \delta r^{c\min(x,y)}, \quad c > 0,\ (x,y) \in \Omega_0. \tag{4.228}$$

To prove our results, we need a modified version of the Darbo fixed point theorem.

Lemma 4.39. *Let Ω be a nonempty, bounded, convex, and closed subset of a Banach space X. If $F:\ \Omega \to \Omega$ is a μ-contraction, then F has at least one fixed point in Ω and the set Fix $F = \{x \in \Omega \mid Fx = x\}$ belongs to the* ker μ.

Remark 4.40. Noted set Fix F with K, it is easy to see that $\mu(K) = \mu(FK) = 0$.

Denote $C_0 = C(\Omega_1, R^n)$, the space of bounded continuous functions on Ω_1 with the norm $\|Z\|_{\Omega_1} = \{\sup \|Z(x,y)\| : (x,y) \in \Omega_1\} < \infty$. So C_0 is a Banach space.

Denote C_H as an arbitrary nonempty and bounded subsets of C_0 such that $\|Z\|_{\Omega_1} \le H$. For any $T > 0$, $\epsilon > 0$, $P = (x_1, y_1)$, $Q = (x_2, y_2) \in [-p, \infty) \times [-q, \infty)$, we denote

$$\begin{aligned}
&w_\epsilon^T(Z) \\
&\quad = \{\sup \|Z(x_1, y_1) - Z(x_2, y_2)\| : P, Q \in [-p, T] \times [-q, T],\ \|P - Q\| \le \epsilon\}, \\
&w_\epsilon^T(C_H) = \{\sup w_\epsilon^T(Z) : Z \in C_H\}, \\
&w_o^T(C_H) = \lim_{\epsilon \to 0} w_\epsilon^T(C_H), \\
&w_o(C_H) = \lim_{T \to \infty} w_o^T(C_H), \\
&a_o(C_H) = \lim_{T \to \infty} \sup_{Z \in C_H} \{\sup \|Z(x_1, y_1)\|,\ P \in [T, \infty) \times [T, \infty)\}, \\
&\mu(C_H) = w_o(C_H) + a_o(C_H).
\end{aligned} \tag{4.229}$$

Similar to the related result in [15], it is not difficult to prove the following conclusion.

Lemma 4.41. *The function $\mu(C_H)$ is the sublinear measure of noncompactness in the space C_0.*

Theorem 4.42. *Suppose the following conditions hold:*

(i) $r = \sup_{x \ge 0,\, y \ge 0} \sum_{k=1}^{N} \|A_k(x, y)\| < 1$; $A_K(x, y)$ *is continuous,* $(x, y) \in \Omega_0$,

(ii) $\lim_{x \to \infty} x - p(x) = \infty$,

(iii) $\lim_{y \to \infty} y - q(y) = \infty$.

Then for every given $\varphi(x, y)$, such that $\sup \|\varphi(x, y)\| = c < +\infty$, $(x, y) \in \Omega_2$, *the corresponding solution $Z(x, y)$ of* (4.223) *satisfies* $\|Z(x, y)\| \to 0$, *as* $\min(x, y) \to +\infty$.

Proof. For any $M > 0$, let $h_M = \{Z \in C_0 : Z(x, y) = \varphi(x, y),\ (x, y) \in \Omega_2$, and $\|Z\|_{\Omega_1} \le M\}$, and $F : h_M \to C_0$ is the map given by

$$(FZ)(x, y) = \begin{cases} \displaystyle\sum_{k=1}^{N} A_k(x, y) Z(x - p_k(x), y - q_k(y)), & (x, y) \in \Omega_0, \\ \varphi(x, y), & (x, y) \in \Omega_2. \end{cases} \tag{4.230}$$

First, we should verify $F(h_M) \subseteq h_M$. For all $Z \in h_M$, we have

$$\begin{aligned}||(FZ)(x,y)|| &= \left|\left|\sum_{k=1}^{N} A_k(x,y)Z(x-p_k(x),y-q_k(y))\right|\right| \\ &\leq \sum_{k=1}^{N} ||A_k(x,y)||||Z(x-p_k(x),y-q_k(y))|| \\ &\leq r\|Z\|_{\Omega_1}.\end{aligned} \tag{4.231}$$

Therefore, $\|FZ\|_{\Omega_1} \leq \|Z\|_{\Omega_1} \leq M$, that is, $F(h_M) \subseteq h_M$.
By the similar way, we obtain

$$||FZ_1 - FZ_2|| \leq ||Z_1 - Z_2||. \tag{4.232}$$

Hence F is continuous.

Next, we should verify $\mu(Fh_M) < \mu(h_M)$.

For any $Z \in h_M,\ T \geq 0,\ x,y \in [T,\infty)\times[T,\infty)$, we can get that $a_0(Fh_M) \leq ra_0(h_M)$. Now, let us take $P = (x_1,y_1),\ Q = (x_2,y_2),\ T > 0,\ P,Q \in (0,T)\times(0,T)$. Since $A_k(x,y)$ is a continuous function, so for any $\varepsilon > 0$, there is $\delta > 0$ such that if $\|P-Q\| \leq \delta$, we have $\|A_k(x_1,y_1) - A_k(x_2,y_2)\| \leq r\varepsilon/MN$. Hence

$$\begin{aligned}&||(FZ)(x_1,y_1) - (FZ)(x_2,y_2)|| \\ &\quad= \left|\left|\sum_{k=1}^{N} A_k(x_1,y_1)Z(x_1-p_k(x_1),y_1-q_k(y_1)) \right.\right.\\ &\qquad\qquad \left.\left. - \sum_{k=1}^{N} A_k(x_2,y_2)Z(x_2-p_k(x_2),y_2-q_k(y_2))\right|\right| \\ &\quad\leq \sum_{k=1}^{N} ||A_k(x_1,y_1)||\,||Z(x_1-p_k(x_1),y_1-q_k(y_1)) \\ &\qquad\qquad\qquad - Z(x_2-p_k(x_2),y_2-q_k(y_2))|| \\ &\qquad + \sum_{k=1}^{N} ||A_k(x_2,y_2) - A_k(x_1,y_1)||\,||Z(x_2-p_k(x_2),y_2-q_k(y_2))|| \\ &\quad\leq \sup\sum_{k=1}^{N} ||A_k(x_1,y_1)||\sup||Z(x_1-p_k(x_1),y_1-q_k(y_1)) \\ &\qquad\qquad\qquad - Z(x_2-p_k(x_2),y_2-q_k(y_2))|| + r\varepsilon \\ &\quad\leq r(\varepsilon + \sup\|Z(x_1-p_k(x_1),y_1-q_k(y_1)) - Z(x_2-p_k(x_2),y_2-q_k(y_2))\|).\end{aligned} \tag{4.233}$$

So, we have

$$
\begin{aligned}
w_\epsilon^T(Fh_M) &\le r(\epsilon + w_\epsilon^T(h_M)),\\
w_o^T(Fh_M) &= \lim_{\epsilon\to 0} w_\epsilon^T(Fh_M) \le r w_o^T(h_M).
\end{aligned}
\tag{4.234}
$$

Since $\mu(h_M) = w_o(h_M) + a_0(h_M)$, $\mu(Fh_M) = w_o(Fh_M) + a_0(Fh_M)$, we can obtain

$$
\mu(Fh_M) \le r\mu(h_M), \tag{4.235}
$$

which means that F is μ-contraction, and by Lemma 4.39, F has a fixed point $Z \in h_M$. It is easy to see that $Z(x,y)$ is a solution of (4.223).

Since $Z \in K$, $\mu(K) = 0$, we get that $a_0(K) = 0$, that is, $\|Z(x,y)\| \to 0$, as $\min(x,y) \to +\infty$.

The proof is complete. □

In Theorem 4.42, we obtain sufficient conditions of the attractivity of the solution of (4.223). In order to reach the conclusion that (4.223) is asymptotically stable in the large, we need to prove that (4.223) is stable.

Theorem 4.43. *Assume the conditions of Theorem 4.42 hold. Then for every given* $\varphi(x,y)$, *such that* $\sup\|\varphi(x,y)\| = c < +\infty, (x,y) \in \Omega_2$, *the solution* $Z(x,y)$ *of* (4.223) *satisfies* $\|Z(x,y)\| \le c$, $(x,y) \in \Omega_0$. *Therefore,* (4.223) *is stable.*

Proof. First we define a sequence of sets S_i in Ω_0 as follows.

For a point $(x,y) \in \Omega_0$, if $(x - p_k(x), y - q_k(y)) \in \Omega_2$, $k = 1,\dots,n$, then $(x,y) \in S_1$.

And for another point $(x,y) \in \Omega_0\backslash S_1$, if $(x - p_k(x), y - q_k(y)) \in \Omega_2 \cup S_1$, $k = 1,\dots,n$, then $(x,y) \in S_2$.

Step by step, we get a series of set $S_1, S_2, S_3, \dots$. We will show that $\Omega_0 = \bigcup_{i=1}^{\infty} S_i$.

In fact, because $p_k(x)$, $q_k(y)$ are both continuous, for any arbitrary point $(x_1,y_1) \in \Omega_0$, there exist two constants $a > 0$, $b > 0$ such that $p_k(x) \ge a$, $q_k(y) \ge b$, $0 \le x \le x_1$ and $0 \le y \le y_1$. It is sure that

$$
(x_1,y_1) \in \bigcup_{i=1}^{\max([x_1/a],[y_1/b])+1} S_i. \tag{4.236}
$$

It is easy to see that

$$
||Z(x,y)|| \le \sum_{k=1}^{n} ||A_k(x,y)||\,||Z(x - p_k(x), y - q_k(y))||. \tag{4.237}
$$

Therefore, $\sup_{(x,y)\in S_1}\|Z(x,y)\| \le c$.

By a similar way, we have

$$\sup_{(x,y)\in S_2} ||Z(x,y)|| \le \max\left(\sup_{(x,y)\in S_1} ||Z(x,y)||, c\right) \le c. \tag{4.238}$$

By induction, we have

$$\sup_{(x,y)\in S_i} ||Z(x,y)|| \le c, \quad i = 1,2,3,\dots. \tag{4.239}$$

The proof is complete. □

Combining Theorems 4.42 and 4.43, we have the following corollary.

Corollary 4.44. *Assume that the assumptions of Theorem 4.42 hold. Then* (4.223) *is asymptotically stable in the large.*

About the exponential asymptotical stability of (4.223), we have the following result.

Theorem 4.45. *Suppose* $r = \sup_{x\ge 0,\, y\ge 0} \sum_{k=1}^{N} \|A_k(x,y)\| < 1$. *If there exist positive numbers* a *and* A *such that* $0 < a \le p_k(x)$, $0 < a \le q_k(y)$; $1 \le k \le N$ *and* $p(x) \le A$, $q(y) \le A$, $(x,y) \in \Omega_0$, *then for every given* $\varphi(x,y) \in R^n$, *with* $\|\varphi\|_{\Omega_2} = \sup_{(x,y)\in\Omega_2} \|\varphi(x,y)\| = c < +\infty$, (4.223) *has a unique solution* $Z(x,y)$ *such that*

$$||Z(x,y)|| \le cr^{[\min(x,y)/A]}, \quad (x,y) \in \Omega_0, \tag{4.240}$$

where $[\cdot]$ *denotes the greatest integer function less than or equal* $\min(x,y)/A$.

Proof. For any given $\varphi(x,y)$, it is easy to see that (4.223) has a unique solution $Z(x,y)$.

First, we assume that $x \le a$ or $y \le a$.

Because $p_k(x) \ge a$, $q_k(y) \ge a$, $1 \le k \le N$, we have $(x - p_k(x), y - q_k(y)) \in \Omega_2$, therefore,

$$\begin{aligned} ||Z(x,y)|| &= \left|\left| \sum_{k=1}^{N} A_k(x,y) Z(x - p_k(x), y - q_k(y)) \right|\right| \\ &\le \sum_{k=1}^{N} ||A_k(x,y)|| \max_{1\le k\le N} ||Z(x - p_k(x), y - q_k(y))|| \\ &\le r \max_{1\le k\le N} ||Z(x - p_k(x), y - q_k(y))|| \\ &\le r\|\varphi\|_{\Omega_2} \le cr. \end{aligned} \tag{4.241}$$

Because $\min(x - p_k(x), y - q_k(y)) \le 0$, (4.240) holds.

Next, we assume for a positive integer m, (4.240) holds for $x \le (m-1)a$ or $y \le (m-1)a$, that is,

$$\|Z(x,y)\| \le cr^{[\min(x,y)/A]}, \quad x \le (m-1)a, \quad \text{or} \quad y \le (m-1)a. \tag{4.242}$$

For $(x,y) \in \{(x,y) \mid x > (m-1)a,\ y > (m-1)a\} \setminus \{(x,y) \mid x > ma,\ y > ma\}$, it is easy to see that $(x - p_k(x)) \le (m-1)a$ or $(y - q_k(y)) \le (m-1)a$.

From (4.241), we have

$$\begin{aligned}\|Z(x,y)\| &\le r \max_{1\le k\le N} \|Z(x-p_k(x), y-q_k(y))\| \\ &\le r \max_{1\le k\le N} cr^{[\min(x-p_k(x),y-q_k(y))/A]} \\ &\le c \max_{1\le k\le N} r^{[\min(x-p_k(x),y-q_k(y))/A+1]}.\end{aligned} \tag{4.243}$$

Since $p_k(x) \le p(x) \le A$, $q_k(y) \le q(y) \le A$, and $\min(x-p_k(x), y-q_k(y)) + A \ge \min(x,y)$, from (4.243), we get

$$\|Z(x,y)\| \le cr^{[\min(x,y)/A]}. \tag{4.244}$$

By the induction, we obtain that (4.240) holds on Ω_0. The proof is complete. □

By Definition 4.38, we obtain the following corollary.

Corollary 4.46. *Under conditions of Theorem 4.45,* (4.223) *is exponentially asymptotically stable.*

Consider the scalar partial difference equation

$$z(x+1,y+1) = a(x,y)z(x+1,y) + b(x,y)z(x,y+1) + p(x,y)z(x-s,y-t) \tag{4.245}$$

for $x \ge 0$, $y \ge 0$, where $s,t > 0$ are constants.

Similar to the proof of Theorem 4.45, we can obtain the following result about the attractivity of solutions of (4.245).

Corollary 4.47. *Assume that* $|a(x,y)| + |b(x,y)| + |p(x,y)| \le r < 1$, *then for any* $\varphi(x,y)$, $(x,y) \in \Omega$, $\Omega = \{(x,y) \mid x \ge -s,\ y \ge -t\} \setminus \{(x,y) \mid x \ge 0,\ y \ge 0\}$ *satisfies* $\sup_{(x,y)\in\Omega} |\varphi(x,y)| = c < +\infty$, (4.245) *has a unique solution* $z(x,y)$ *with* $|z(x,y)| \to 0$, *as* $\min(x,y) \to +\infty$. *Hence,* (4.245) *is attractive.*

If we put more conditions on the initial function, then we can obtain stronger results.

Theorem 4.48. *Suppose* $r = \sup_{x\geq 0,\, y\geq 0} \sum_{k=1}^{N} \|A_k(x,y)\| < 1$. *If there exist positive numbers a and A such that* $0 < a \leq p_k(x) + q_k(y)$, $1 \leq k \leq N$ *and* $p(x) + q(y) \leq A$, $(x,y) \in \Omega_0$, *then for every given* $\varphi(x,y) \in R^n$, *with* $\|\varphi(x,y)\| \leq cr^{(x+y)/A}$ *on* Ω_2, (4.223) *has a unique solution* $Z(x,y)$ *such that*

$$||Z(x,y)|| \leq cr^{[(x+y)/A]}, \quad (x,y) \in \Omega_0, \tag{4.246}$$

where $[\cdot]$ *denotes the greatest integer function less than or equal* $(x+y)/A$.

Proof. For any given $\varphi(x,y)$, it is easy to see that (4.223) has a unique solution $Z(x,y)$.

First, we assume that $x + y \leq a$.

Because $p_k(x) + q_k(y) > a$, $1 \leq k \leq N$, we have $(x - p_k(x), y - q_k(y)) \in \Omega_2$, therefore,

$$\begin{aligned} ||Z(x,y)|| &= \left|\left| \sum_{k=1}^{N} A_k(x,y) Z(x - p_k(x), y - q_k(y)) \right|\right| \\ &\leq \sum_{k=1}^{N} ||A_k(x,y)|| \max_{1\leq k\leq N} ||Z(x - p_k(x), y - q_k(y))|| \\ &\leq r \max_{1\leq k\leq N} ||Z(x - p_k(x), y - q_k(y))|| \\ &\leq r\|\varphi\|_{\Omega_2} \leq cr. \end{aligned} \tag{4.247}$$

Because $[(x+y)/A] = 0$, (4.246) holds.

Next, we assume for a positive integer m, (4.246) holds for $x + y \leq (m-1)a$, that is,

$$||Z(x,y)|| \leq cr^{[(x+y)/A]}, \quad x + y \leq (m-1)a. \tag{4.248}$$

For $(m-1)a < x + y \leq ma$, $(x - p_k(x)) + (y - q_k(y)) \leq (m-1)a$, from (4.247), we have,

$$\begin{aligned} ||Z(x,y)|| &\leq r \max_{1\leq k\leq N} ||Z(x - p_k(x), y - q_k(y))|| \\ &\leq r \max_{1\leq k\leq N} cr^{[(x+y-p_k(x)-q_k(y))/A]} \\ &\leq c \max_{1\leq k\leq N} r^{[(x+y-p_k(x)-q_k(y))/A+1]}. \end{aligned} \tag{4.249}$$

By the induction, we obtain that (4.246) holds on Ω_0. The proof is complete. □

Example 4.49. Consider the system of delay partial difference equations

$$Z(x,y) = A_1(x,y) Z(x - p_1(x), y - q_1(y)) + A_2(x,y) Z(x - p_2(x), y - q_2(y)), \tag{4.250}$$

where $Z \in R^2$,

$$A_1(x,y) = \begin{pmatrix} \dfrac{\sin(xy)}{3} & 0 \\ 0 & \dfrac{-\sin(xy)}{3} \end{pmatrix}, \qquad A_2(x,y) = \begin{pmatrix} 0 & \dfrac{\sin(x+y)}{3} \\ \dfrac{-\sin(x+y)}{3} & 0 \end{pmatrix}. \tag{4.251}$$

It is easy to check that

$$\begin{gathered} \|A_1\| = \|A_2\| = \frac{1}{3}, \\ r = \|A_1\| + \|A_2\| = \frac{2}{3} < 1. \end{gathered} \tag{4.252}$$

First, we suppose $p_1(x) = 2$, $q_1(y) = 3$, $p_2(x) = 1$, $q_2(y) = 4$.
Let $a = 0.5$, $A = 4$, and

$$\begin{gathered} \Omega_0 = \{(x,y) \mid x > 0,\ y > 0\}, \\ \Omega_1 = \{(x,y) \mid x \geq -2,\ y \geq -4\}, \\ \Omega_2 = \Omega_1 \backslash \Omega_0. \end{gathered} \tag{4.253}$$

From Theorem 4.45, we obtain the following conclusion.

Given any initial function $\varphi(x,y) \in R^2$, and $\|\varphi\|_{\Omega_2} = c < +\infty$, there exists a solution $Z(x,y)$ of (4.250) with

$$\|Z(x,y)\| \leq c r^{[\min(x,y)/4]}, \quad (x,y) \in \Omega_0. \tag{4.254}$$

Next, we suppose $p_1(x) = 0.5x + 2$, $q_1(y) = \ln y - 3$, $p_2(x) = (13/x) + 1$, $q_2(y) = 4$, where $p_1(x)$, $p_2(x)$, $q_1(y)$ are unbounded on Ω_0.

From Theorem 4.42, for any given function $\varphi(x,y) \in R^2$, and $\|\varphi\|_{\Omega_2} = c < +\infty$, there exists a solution $Z(x,y)$ of (4.250) with

$$\|Z(x,y)\| \longrightarrow 0, \quad \text{when } \min(x,y) \longrightarrow +\infty. \tag{4.255}$$

4.4. Stability of discrete delay logistic equations

4.4.1. Stability of 2D discrete logistic system

In engineering applications, particularly in the fields of digital filtering, imaging, and spatial dynamical systems, 2D discrete systems have been a focal subject for investigation.

Consider the delayed 2D discrete logistic system

$$x_{m+1,n} + a_{m,n} x_{m,n+1} = \mu_{m,n} x_{m,n} (1 - x_{m-\sigma,n-\tau}), \tag{4.256}$$

where $\{a_{m,n}\}$ and $\{\mu_{m,n}\}$ are two double sequences of real numbers, σ and τ are nonnegative integers, and $m, n \in N_0$. The stability and exponential stability of system (4.256) are important properties; in this section, some sufficient conditions for the stability and exponential stability of system (4.256) are derived.

First, observe that in the special case where $a_{m,n} = a$, $\mu_{m,n} = \mu$ and $\sigma = \tau = 0$, system (4.256) becomes

$$x_{m+1,n} + ax_{m,n+1} = \mu x_{m,n}(1 - x_{m,n}) \tag{4.257}$$

and, when $a \equiv 0$ and $n = n_0$, system (4.257) further reduces to

$$x_{m+1,n_0} = \mu x_{m,n_0}(1 - x_{m,n_0}), \tag{4.258}$$

which is just the familiar simple case of the 1D logistic system. Therefore, system (4.256) is quite general.

Let $\Omega = N_{-\sigma} \times N_{-\tau} \setminus N_1 \times N_0$. Obviously, for any given initial function $\varphi = \{\varphi_{m,n}\}$ defined on Ω, by iteration, it is easy to construct via induction a double sequence $\{x_{m,n}\}$ that equals initial conditions $\varphi_{m,n}$ on Ω and satisfies (4.256) for $m, n = 0, 1, 2, \ldots$. Indeed, one can rewrite system (4.256) as

$$x_{m+1,n} = \mu_{m,n}x_{m,n}(1 - x_{m-\sigma,n-\tau}) - a_{m,n}x_{m,n+1} \tag{4.259}$$

and then use it to calculate, successively, $x_{1,0}, x_{1,1}, x_{2,0}, x_{1,2}, x_{2,1}, x_{3,0}, \ldots$.

Let $\|\varphi\| = \sup\{|\varphi_{m,n} \mid (m, n) \in \Omega\}$, δ a positive constant, and

$$S_\delta = \{\varphi \mid \|\varphi\| < \delta\}. \tag{4.260}$$

The definitions of the stability, linear stability, and exponential asymptotical stability of system (4.256) are similar to those in Section 4.2. Now we give a more general stability definition as follows.

Definition 4.50. If there exists a positive number $M > 0$ such that for any constant $\delta \in (0, M)$, there exists a constant $\xi \in (0, 1)$ such that for any given bounded function $\varphi = \{\varphi_{m,n}\}$ defined on $\Omega, \varphi \in S_\delta$ implies that the solution $\{x_{m,n}\}$ of system (4.256) with the initial condition φ satisfies

$$|x_{m,n}| < M\xi^{m+n}, \quad (m, n) \in N_1 \times N_0, \tag{4.261}$$

then system (4.256) is said to be double-variable-bounded-initial exponentially stable, or D-B-exponentially stable.

Obviously, if system (4.256) is exponentially asymptotically stable, then it is stable. If system (4.256) is D-B-exponentially stable, then it is exponentially asymptotically stable and thus it is also stable.

Theorem 4.51. *Assume that there exist two constants, $\alpha > 0$, $C > 0$, and a positive integer k such that*

$$|a_{m,n}| + |\mu_{m,n}| \le C \quad \forall m \in \{0,1,2,\dots,k\} \text{ and any } n \in N_0, \tag{4.262}$$

$$|a_{m,n}| + |\mu_{m,n}|(1+\alpha) \le 1 \quad \forall (m,n) \in N_{k+1} \times N_0. \tag{4.263}$$

Then system (4.256) *is stable.*

Proof. In view of (4.262), it is obvious that there exist two constants $D > C$ and $M > D^{k+1} > 1$ such that

$$|a_{m,n}| + |\mu_{m,n}|(1+\alpha) \le D \quad \forall m \in \{0,1,2,\dots,k\} \text{ and any } n \in N_0. \tag{4.264}$$

For any small constant $\varepsilon > 0$, without loss of generality, assume that $\varepsilon \le \alpha/M$, and let $\delta = \varepsilon$ and let $\varphi \in S_\delta$ be a bounded function defined on Ω. We claim that the solution $\{x_{m,n}\}$ of system (4.256) with the initial condition φ satisfies $|x_{m,n}| < \varepsilon$ for $(m,n) \in N_0^2$. In fact, from (4.256), (4.259), and (4.264), we have

$$\begin{aligned} |x_{1,0}| &= |\mu_{0,0}x_{0,0}(1 - x_{-\sigma,-\tau}) - a_{0,0}x_{0,1}| \le (|a_{0,0}| + |\mu_{0,0}|(1+\alpha))\delta \le D\varepsilon \le \alpha, \\ |x_{1,1}| &= |\mu_{0,1}x_{0,1}(1 - x_{-\sigma,1-\tau}) - a_{0,1}x_{0,2}| \le (|a_{0,1}| + |\mu_{0,1}|(1+\alpha))\delta \le D\varepsilon \le \alpha. \end{aligned} \tag{4.265}$$

Assume that for a certain integer $m \in \{1,2,\dots,k\}$,

$$|x_{i,n}| \le D^i\varepsilon \le \alpha \quad \forall 1 \le i \le m,\ n \ge 0. \tag{4.266}$$

Then, from (4.259) and (4.264),

$$\begin{aligned} |x_{m+1,n}| &= |\mu_{m,n}x_{m,n}(1 - x_{m-\sigma,n-\tau}) - a_{m,n}x_{m,n+1}| \\ &\le (|a_{m,n}| + |\mu_{m,n}|(1+\alpha))D^m\varepsilon \le D^{m+1}\varepsilon. \end{aligned} \tag{4.267}$$

Hence, $|x_{m,n}| \le M\varepsilon$ for all $m \in \{0,1,2,\dots,k+1\}$ and all $n \in N_0$.

Assume that for a certain $m \ge k+1$,

$$|x_{i,n}| \le M\varepsilon \le \alpha \quad \forall 1 \le i \le m,\ n \ge 0. \tag{4.268}$$

Then, from (4.256) and (4.263),

$$\begin{aligned} |x_{m+1,n}| &= |\mu_{m,n}x_{m,n}(1 - x_{m-\sigma,n-\tau}) - a_{m,n}x_{m,n+1}| \\ &\le (|a_{m,n}| + |\mu_{m,n}|(1+\alpha))M\varepsilon \le M\varepsilon \le \alpha. \end{aligned} \tag{4.269}$$

Hence, by induction, $|x_{m,n}| \le M\varepsilon$ for all $(m,n) \in N_1 \times N_0$, that is, system (4.256) is stable. The proof is complete. □

Next, let $A_{m,n} = |a_{m,n}| + |\mu_{m,n}|$ for all $(m,n) \in N_0^2$, and

$$\bar{a}_{m,n} = a_{m,n}A_{m-1,n+1}, \quad \bar{\mu}_{m,n} = \mu_{m,n}A_{m-1,n} \quad \forall (m,n) \in N_1 \times N_0. \tag{4.270}$$

Theorem 4.52. *Assume that there exist constants* $\alpha > 0$ *and* $C > 0$ *such that*

$$|a_{0,n}| + |\mu_{0,n}| \le C, \quad |\bar{a}_{1,n}| + |\bar{\mu}_{1,n}| \le C \quad \forall n \in N_0, \tag{4.271}$$

and for any $(m,n) \in N_2 \times N_0$,

$$\begin{aligned} &|a_{m,n}|\{|a_{m-1,n+1}| + |\mu_{m-1,n+1}|(1+\alpha)\} \\ &\quad + |\mu_{m,n}|\{|a_{m-1,n}| + |\mu_{m-1,n}|(1+\alpha)\}(1+\alpha) \le 1. \end{aligned} \tag{4.272}$$

Then, system (4.256) *is stable.*

Proof. From (4.271) and (4.272), there exists a constant $M \in [1+C,\infty)$ such that

$$\begin{gathered} |a_{0,n}| + |\mu_{0,n}|(1+\alpha) \le M, \\ |a_{1,n}|\{|a_{0,n+1}| + |\mu_{0,n+1}|(1+\alpha)\} + |\mu_{1,n}|\{|a_{0,n}| + |\mu_{0,n}|(1+\alpha)\}(1+\alpha) \le M \end{gathered} \tag{4.273}$$

for all $n \in N_0$.

For any small constant $\varepsilon \in (0, \alpha/M)$, let $\delta = \varepsilon$ and $\varphi \in S_\delta$ be a bounded function defined on Ω. Let $\{x_{m,n}\}$ be a solution of system (4.256) with the initial condition φ. Then, $|x_{m,n}| < \delta$ for all $(m,n) \in \Omega$. In view of (4.256), for any nonnegative integer n, we have

$$\begin{aligned} |x_{1,n}| &= |\mu_{0,n}x_{0,n}(1 - x_{-\sigma,n-\tau}) - a_{0,n}x_{0,n+1}| \\ &\le (|a_{0,n}| + (1+\alpha)|\mu_{0,n}|) \cdot \delta \le M\varepsilon \le \alpha. \end{aligned} \tag{4.274}$$

From (4.256), for all $m \ge 1$ and all $n \in N_0$,

$$\begin{aligned} x_{m+1,n} &= \mu_{m,n}x_{m,n}(1 - x_{m-\sigma,n-\tau}) - a_{m,n}x_{m,n+1} \\ &= \mu_{m,n}(-a_{m-1,n}x_{m-1,n+1} + \mu_{m-1,n}x_{m-1,n}(1 - x_{m-1-\sigma,n-\tau})) \\ &\quad \times (1 - x_{m-\sigma,n-\tau}) - a_{m,n}(-a_{m-1,n+1}x_{m-1,n+2} + \mu_{m-1,n+1}x_{m-1,n+1} \\ &\quad \times (1 - x_{m-1-\sigma,n+1-\tau})). \end{aligned} \tag{4.275}$$

Then, from (4.275),

$$\begin{aligned}|x_{2,n}| &\le |\mu_{1,n}(-a_{0,n}x_{0,n+1}+\mu_{0,n}x_{0,n}(1-x_{-\sigma,n-\tau}))(1-x_{1-\sigma,n-\tau})|\\&\quad+|a_{1,n}(-a_{0,n+1}x_{0,n+2}+\mu_{0,n+1}x_{0,n+1}(1-x_{-\sigma,n+1-\tau}))|\\&\le \{|\mu_{1,n}|(|a_{0,n}|+|\mu_{0,n}|(1+\alpha))(1+\alpha)\\&\quad+|a_{1,n}|(|a_{0,n+1}|+|\mu_{0,n+1}|(1+\alpha))\}\cdot\delta\\&\le M\varepsilon\le\alpha\quad\forall n\in N_0.\end{aligned}\tag{4.276}$$

Assume that for some integer $m \ge 2$,

$$|x_{i,n}| \le M\varepsilon \le \alpha \quad \forall 1\le i\le m \text{ and all } n\in N_0. \tag{4.277}$$

Then, from (4.272) and (4.275),

$$\begin{aligned}|x_{m+1,n}| &\le |\mu_{m,n}|(|a_{m-1,n}|+|\mu_{m-1,n}|(1+\alpha))(1+\alpha)\cdot M\varepsilon\\&\quad+|a_{m,n}|(|a_{m-1,n+1}|+|\mu_{m-1,n+1}|(1+\alpha))\cdot M\varepsilon\\&\le M\varepsilon\le\alpha\quad\forall n\in N_0.\end{aligned}\tag{4.278}$$

By induction, $|x_{m,n}| \le M\varepsilon$ for all $(m,n)\in N_1\times N_0$, that is, system (4.256) is stable. The proof is complete. □

Now, let

$$\begin{aligned}D_1&=\{(m,n)\mid 1\le m\le\sigma,\ 0\le n<\tau\}, & D_2&=\{(m,n)\mid m>\sigma,\ 0\le n<\tau\},\\D_3&=\{(m,n)\mid 1\le m\le\sigma,\ n\ge\tau\}, & D_4&=\{(m,n)\mid m>\sigma,\ n\ge\tau\}.\end{aligned}\tag{4.279}$$

Obviously, D_1, D_2, D_3, D_4 are disjoint of one another, and $N_1\times N_0 = D_1\cup D_2\cup D_3\cup D_4$.

Theorem 4.53. *Assume that there exist constants $M>0$ and $\xi\in(0,1)$ such that*

$$\begin{gathered}|a_{0,n}|+(1+M)|\mu_{0,n}|\le\xi^{n+1}\quad\forall n\in N_0,\\|a_{m,n}|+|\mu_{m,n}|(1+M)\xi^{-1}\le 1\quad\forall (m,n)\in D_1\cup D_2\cup D_3,\end{gathered}\tag{4.280}$$

$$|a_{m,n}|+|\mu_{m,n}|(1+M\xi^{m+n-\sigma-\tau})\xi^{-1}\le 1\quad\forall (m,n)\in D_4.\tag{4.281}$$

Then, system (4.256) *is D-B-exponentially stable.*

Proof. For any constant $\delta \in (0, M)$ and any bounded function $\varphi \in S_\delta$, let $\{x_{m,n}\}$ be a solution of system (4.256) with initial condition φ. From (4.256) and (4.280), for the given constant $\xi \in (0,1)$, we have

$$\begin{aligned} |x_{1,n}| &= |\mu_{0,n}x_{0,n}(1 - x_{-\sigma,n-\tau}) - a_{0,n}x_{0,n+1}| \\ &\leq |\mu_{0,n}|\delta(1+\delta) + |a_{0,n}|\delta \leq M\xi^{n+1}, \quad n \in N_0. \end{aligned} \tag{4.282}$$

Assume that for a certain $m \in \{1, 2, \ldots, \sigma\}$,

$$|x_{i,j}| \leq M\xi^{i+j} \quad \forall 1 \leq i \leq m \text{ and all } j \in N_0. \tag{4.283}$$

Then, for all $n \geq 0$, one has $(m-\sigma, n-\tau) \in \Omega$ and $(m, n) \in D_1 \cup D_2 \cup D_3$. Hence, from (4.256) and (4.281), we have

$$\begin{aligned} |x_{m+1,n}| &\leq |\mu_{m,n}| \cdot |x_{m,n}|(1 + |x_{m-\sigma,n-\tau}|) + |a_{m,n}| \cdot |x_{m,n+1}| \\ &\leq M\xi^{m+n+1}\{|\mu_{m,n}|(1+\delta)\xi^{-1} + |a_{m,n}|\} \leq M\xi^{m+n+1}. \end{aligned} \tag{4.284}$$

By induction, $|x_{m,n}| \leq M\xi^{m+n}$ for all $m \in \{1, 2, \ldots, \sigma + 1\}$ and all $n \geq 0$.

Assume that for a certain $m \in N_{\sigma+1}$,

$$|x_{i,n}| \leq M\xi^{i+n} \leq M \quad \forall 1 \leq i \leq m \text{ and all } n \in N_0. \tag{4.285}$$

Then, if $n \in \{0, 1, \ldots, \tau - 1\}$, one has $(m, n) \in D_1 \cup D_2 \cup D_3$ and $(m-\sigma, n-\tau) \in \Omega$. From (4.256) and (4.281),

$$\begin{aligned} |x_{m+1,n}| &\leq |\mu_{m,n}| \cdot |x_{m,n}|(1 + |x_{m-\sigma,n-\tau}|) + |a_{m,n}| \cdot |x_{m,n+1}| \\ &\leq M\xi^{m+n+1}\{|\mu_{m,n}|(1+\delta)\xi^{-1} + |a_{m,n}|\} \leq M\xi^{m+n+1}, \end{aligned} \tag{4.286}$$

if $n \geq \tau$, then $(m, n) \in D_4$ and $(m-\sigma, n-\tau) \in N_1 \times N_0$. From (4.256) and (4.281), by the assumption of induction, we obtain

$$\begin{aligned} |x_{m+1,n}| &\leq |\mu_{m,n}| \cdot |x_{m,n}|(1 + |x_{m-\sigma,n-\tau}|) + |a_{m,n}| \cdot |x_{m,n+1}| \\ &\leq M\xi^{m+n+1}\{|\mu_{m,n}|(1 + M\xi^{m+n-\sigma-\tau})\xi^{-1} + |a_{m,n}|\} \leq M\xi^{m+n+1}. \end{aligned} \tag{4.287}$$

By induction, $|x_{m,n}| \leq M\xi^{m+n}$ for all $(m, n) \in N_1 \times N_0$, that is, system (4.256) is D-B-exponentially stable. The proof is complete. □

In the following, let $D_0 = \{(m, n) \mid m = 1,\ n \geq 0\}$,

$$\begin{aligned} \overline{D}_0 &= \{(m,n) \mid m = \sigma+1,\ 0 \leq n < \tau\}, & \widetilde{D}_0 &= \{(m,n) \mid m = \sigma+1,\ n \geq \tau\}, \\ \overline{D}_1 &= \{(m,n) \mid 2 \leq m \leq \sigma,\ 0 \leq n < \tau\}, & \overline{D}_2 &= \{(m,n) \mid m > \sigma+1,\ 0 \leq n < \tau\}, \\ \overline{D}_3 &= \{(m,n) \mid 2 \leq m \leq \sigma,\ n \geq \tau\}, & \overline{D}_4 &= \{(m,n) \mid m > \sigma+1,\ n \geq \tau\}. \end{aligned} \tag{4.288}$$

Obviously, $D_0, \overline{D}_1, \overline{D}_2, \overline{D}_3, \overline{D}_0, \widetilde{D}_0, \overline{D}_4$ are disjoint of one another, and

$$N_1 \times N_0 = D_0 \cup \overline{D}_1 \cup \overline{D}_2 \cup \overline{D}_3 \cup \overline{D}_0 \cup \widetilde{D}_0 \cup \overline{D}_4. \tag{4.289}$$

Theorem 4.54. *Assume that $\sigma > 0$, and there exist constants $M > 0$ and $\xi \in (0,1)$ such that*

(i) *for all $n \in N_0$,*

$$|a_{0,n}| + (1+M)|\mu_{0,n}| \le \xi^{n+1}, \tag{4.290}$$

(ii) *for all $(m,n) \in D_0$,*

$$\begin{aligned} &|a_{m,n}|\big(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|(1+M)\big) \\ &\quad + |\mu_{m,n}|\big(|a_{m-1,n}| + |\mu_{m-1,n}|(1+M)\big)(1+M) \le \xi^{n+2}, \end{aligned} \tag{4.291}$$

(iii) *for all $(m,n) \in \overline{D}_1 \cup \overline{D}_2 \cup \overline{D}_3 \cup \overline{D}_0$,*

$$\begin{aligned} &|a_{m,n}|\big(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|(1+M)\xi^{-1}\big) \\ &\quad + |\mu_{m,n}|\big(|a_{m-1,n}| + |\mu_{m-1,n}|(1+M)\xi^{-1}\big)(1+M)\xi^{-1} \le 1, \end{aligned} \tag{4.292}$$

(iv) *for all $(m,n) \in \widetilde{D}_0$,*

$$\begin{aligned} &|a_{m,n}|\big(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|(1+M)\xi^{-1}\big) \\ &\quad + |\mu_{m,n}|\big(|a_{m-1,n}| + |\mu_{m-1,n}|(1+M)\xi^{-1}\big)\big(1+M\xi^{m+n-\sigma-\tau}\big)\xi^{-1} \le 1, \end{aligned} \tag{4.293}$$

(v) *for all $(m,n) \in \overline{D}_4$,*

$$\begin{aligned} &|a_{m,n}|\big(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|\big(1+M\xi^{m+n-\sigma-\tau}\big)\xi^{-1}\big) \\ &\quad + |\mu_{m,n}|\big(|a_{m-1,n}| + |\mu_{m-1,n}|\big(1+M\xi^{m+n-1-\sigma-\tau}\big)\xi^{-1}\big) \\ &\quad \times \big(1+M\xi^{m+n-\sigma-\tau}\big)\xi^{-1} \le 1. \end{aligned} \tag{4.294}$$

Then system (4.256) *is D-B-exponentially stable.*

Proof. For any constant $\delta \in (0,M)$ and any given function $\varphi \in S_\delta$, let $\{x_{m,n}\}$ be a solution of system (4.256) with initial condition φ. From (4.256) and (4.290), for the given $\xi \in (0,1)$ and any $n \in N_0$, we have

$$\begin{aligned} |x_{1,n}| &= |\mu_{0,n}x_{0,n}(1-x_{-\sigma,n-\tau}) - a_{0,n}x_{0,n+1}| \\ &\le |\mu_{0,n}|\delta(1+\delta) + |a_{0,n}|\delta \le M\xi^{n+1} < M, \end{aligned} \tag{4.295}$$

that is, $|x_{m,n}| \leq M\xi^{m+n}$ for all $(m,n) \in D_0$. Hence, from (4.256), (4.275), and (4.291),

$$\begin{aligned}
|x_{2,n}| &\leq |\mu_{1,n}(-a_{0,n}x_{0,n+1} + \mu_{0,n}x_{0,n}(1 - x_{-\sigma,n-\tau}))(1 - x_{1-\sigma,n-\tau})| \\
&\quad + |a_{1,n}(-a_{0,n+1}x_{0,n+2} + \mu_{0,n+1}x_{0,n+1}(1 - x_{-\sigma,n+1-\tau}))| \\
&\leq \{|\mu_{1,n}| \cdot M(|a_{0,n}| + |\mu_{0,n}|(1+\delta))(1+\delta) \\
&\quad + |a_{1,n}| \cdot M(|a_{0,n+1}| + |\mu_{0,n+1}|(1+\delta))\} \\
&\leq M\xi^{n+2} \quad \forall n \in N_0.
\end{aligned} \tag{4.296}$$

Assume that for a certain $m \in \{2,3,\dots,\sigma\}$,

$$|x_{i,j}| \leq M\xi^{i+j} \quad \forall 1 \leq i \leq m, \text{ and all } j \in N_0. \tag{4.297}$$

Then, for all $n \geq 0$,

$$\begin{gathered}
(m-1,n),(m-1,n+1),(m-1,n+2) \in D_0 \cup \overline{D}_1 \cup \overline{D}_3, \quad (m,n) \in \overline{D}_1 \cup \overline{D}_3, \\
(m-1-\sigma,n-\tau),(m-1-\sigma,n+1-\tau) \in \Omega, \quad (m-\sigma,n-\tau) \in \Omega.
\end{gathered} \tag{4.298}$$

Hence, from (4.256), (4.275), and (4.292), we obtain

$$\begin{aligned}
|x_{m+1,n}| &\leq |\mu_{m,n}|(|a_{m-1,n}| \cdot |x_{m-1,n+1}| + |\mu_{m-1,n}| \cdot |x_{m-1,n}|(1 + |x_{m-1-\sigma,n-\tau}|)) \\
&\quad \times (1 + |x_{m-\sigma,n-\tau}|) + |a_{m,n}|(|a_{m-1,n+1}| \cdot |x_{m-1,n+2}| \\
&\quad + |\mu_{m-1,n+1}| \cdot |x_{m-1,n+1}|(1 + |x_{m-1-\sigma,n+1-\tau}|)) \\
&\leq M\xi^{m+n}|\mu_{m,n}|(|a_{m-1,n}| + |\mu_{m-1,n}|(1+\delta)\xi^{-1})(1+\delta) \\
&\quad + M\xi^{m+n+1}|a_{m,n}|(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|(1+\delta)\xi^{-1}) \\
&\leq M\xi^{m+n+1} \quad \forall n \geq 0.
\end{aligned} \tag{4.299}$$

By induction, $|x_{m,n}| \leq M\xi^{m+n}$ for all $m \in \{1,2,\dots,\sigma+1\}$ and all $n \geq 0$.

From (4.256), (4.275), and (4.292), for all $n \in \{0,1,\dots,\tau-1\}$, one has $(\sigma+1,n) \in \overline{D}_0$ and

$$\begin{aligned}
|x_{\sigma+2,n}| &\leq M\xi^{\sigma+n+1}|\mu_{\sigma+1,n}|(|a_{\sigma,n}| + |\mu_{\sigma,n}|(1+\delta)\xi^{-1})(1+\delta) \\
&\quad + M\xi^{\sigma+n+2}|a_{\sigma+1,n}|(|a_{\sigma,n+1}| + |\mu_{\sigma,n+1}|(1+\delta)\xi^{-1}) \\
&\leq M\xi^{\sigma+n+2}.
\end{aligned} \tag{4.300}$$

From (4.256), (4.275), (4.293), and (4.295), for any $n \geq \tau$, one has $(\sigma+1,n) \in \widetilde{D}_0$ and

$$\begin{aligned} |x_{\sigma+2,n}| &\leq M\xi^{\sigma+n+1}|\mu_{\sigma+1,n}|\left(|a_{\sigma,n}| + |\mu_{\sigma,n}|(1+\delta)\xi^{-1}\right)\left(1+M\xi^{n+1-\tau}\right) \\ &\quad + M\xi^{\sigma+n+2}|a_{\sigma+1,n}|\left(|a_{\sigma,n+1}| + |\mu_{\sigma,n+1}|(1+\delta)\xi^{-1}\right) \\ &\leq M\xi^{\sigma+n+2}. \end{aligned} \tag{4.301}$$

Hence, for all $n \in N_0$, $|x_{\sigma+2,n}| \leq M\xi^{\sigma+n+2}$.

Assume that for a certain $m > \sigma+1$,

$$|x_{i,j}| \leq M\xi^{i+j} \leq M \quad \forall 1 \leq i \leq m \text{ and all } j \in N_0. \tag{4.302}$$

Then, if $n \in \{0,1,\dots,\tau-1\}$, one has $(m,n) \in \overline{D}_2$ and

$$\begin{aligned} &(m-1-\sigma, n-\tau), (m-\sigma, n-\tau) \in \Omega, \\ &(m-1-\sigma, n+1-\tau) \in \Omega \cup \{\{1,2,\dots,m\} \times N_0\}. \end{aligned} \tag{4.303}$$

Hence, from (4.256), (4.275), and (4.292), we have

$$\begin{aligned} |x_{m+1,n}| &\leq M\xi^{m+n}|\mu_{m,n}|\left(|a_{m-1,n}| + |\mu_{m-1,n}|(1+\delta)\xi^{-1}\right)(1+\delta) \\ &\quad + M\xi^{m+n+1}|a_{m,n}|\left(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|(1+M)\xi^{-1}\right) \\ &\leq M\xi^{m+n+1}. \end{aligned} \tag{4.304}$$

if $n \geq \tau$, then

$$(m-1-\sigma, n-\tau), (m-\sigma, n-\tau), (m-1-\sigma, n+1-\tau) \overline{\in} \Omega. \tag{4.305}$$

From (4.275) and (4.294), one has $(m,n) \in \overline{D}_4$ and

$$\begin{aligned} |x_{m+1,n}| &\leq M\xi^{m+n}|\mu_{m,n}|\left(|a_{m-1,n}| + |\mu_{m-1,n}|\left(1+M\xi^{m+n-1-\sigma-\tau}\right)\xi^{-1}\right) \\ &\quad \times \left(1+M\xi^{m+n-\sigma-\tau}\right) + M\xi^{m+n+1}|a_{m,n}| \\ &\quad \times \left(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|\left(1+M\xi^{m+n-\sigma-\tau}\right)\xi^{-1}\right) \\ &\leq M\xi^{m+n+1}. \end{aligned} \tag{4.306}$$

Hence, by induction, $|x_{m,n}| \leq M\xi^{m+n}$ for all $(m,n) \in N_1 \times N_0$. The proof is complete. □

Corollary 4.55. *Assume that $\sigma > 0$, and there exist constants $M > 0$ and $\xi \in (0,1)$ such that*

$$\begin{gathered} |a_{0,n}| + (1+M)|\mu_{0,n}| \le \xi^{n+1} \quad \forall n \in N_0, \\ |a_{1,n}|\left(|a_{0,n+1}| + |\mu_{0,n+1}|(1+M)\right) \\ + |\mu_{0,n}|\left(|a_{0,n}| + |\mu_{0,n}|(1+M)\right)(1+M) \le \xi^{n+2} \end{gathered} \tag{4.307}$$

for all $n \in N_0$, and

$$|\overline{a}_{m,n}| + |\overline{\mu}_{m,n}| \le \xi \quad \forall (m,n) \in N_2 \times N_0, \tag{4.308}$$

where $\overline{a}_{m,n}$ and $\overline{\mu}_{m,n}$ are defined in Theorem 4.52. Then, system (4.256) is D-B-exponentially stable.

Similar to the proof of Theorem 4.54, we can prove the following results.

Theorem 4.56. *Assume that $\sigma = 0$, and there exist constants $M > 0$ and $\xi \in (0,1)$ such that*

(i) *for all $n \in N_0$,*

$$|a_{0,n}| + (1+M)|\mu_{0,n}| \le \xi^{n+1}, \tag{4.309}$$

(ii) *for all $n \in \{0,1,\ldots,\tau-1\}$,*

$$\begin{gathered} |a_{1,n}|\left(|a_{0,n+1}| + |\mu_{0,n+1}|(1+M)\right) \\ + |\mu_{1,n}|\left(|a_{0,n}| + |\mu_{0,n}|(1+M)\right)(1+M) \le \xi^{n+2}, \end{gathered} \tag{4.310}$$

(iii) *for all $n \ge \tau$,*

$$\begin{gathered} |a_{1,n}|\left(|a_{0,n+1}| + |\mu_{0,n+1}|(1+M)\right) \\ + |\mu_{1,n}|\left(|a_{0,n}| + |\mu_{0,n}|(1+M)\right)\left(1+M\xi^{n+1-\tau}\right) \le \xi^{n+2}, \end{gathered} \tag{4.311}$$

(iv) *for all $m \ge 2$ and all $n \in \{0,1,\ldots,\tau-1\}$,*

$$\begin{gathered} |a_{m,n}|\left(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|(1+M)\xi^{-1}\right) \\ + |\mu_{m,n}|\left(|a_{m-1,n}| + |\mu_{m-1,n}|(1+M)\xi^{-1}\right)(1+M)\xi^{-1} \le 1, \end{gathered} \tag{4.312}$$

(v) *for all $m \ge 2$ and all $n \ge \tau$,*

$$\begin{gathered} |a_{m,n}|\left(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|\left(1+M\xi^{m+n-\sigma-\tau}\right)\xi^{-1}\right) \\ + |\mu_{m,n}|\left(|a_{m-1,n}| + |\mu_{m-1,n}|\left(1+M\xi^{m+n-1-\sigma-\tau}\right)\xi^{-1}\right) \\ \times\left(1+M\xi^{m+n-\sigma-\tau}\right)\xi^{-1} \le 1. \end{gathered} \tag{4.313}$$

Then system (4.256) is D-B-exponentially stable.

Theorem 4.57. *Assume that there exists a constant $r \in (0,1)$ such that*

$$\begin{aligned} |a_{0,n}| + |\mu_{0,n}| &\le r \quad \forall n \ge N_0, \\ |\overline{a}_{m,n}| + |\overline{\mu}_{m,n}| &\le r \quad \forall (m,n) \in N_1 \times N_0, \end{aligned} \tag{4.314}$$

where $\overline{a}_{m,n}$ and $\overline{\mu}_{m,n}$ are defined in Theorem 4.52. Then, system (4.256) *is exponentially asymptotically stable.*

Proof. It is obvious that there exist constants $\alpha > 0$ and $R \in (0,1)$ such that $(1+\alpha)^2 r \le R^2$. In view of (4.314), one has

$$|a_{0,n}| + |\mu_{0,n}|(1+\alpha) \le R \quad \forall n \ge N_0, \tag{4.315}$$

and for all $(m,n) \in N_1 \times N_0$,

$$\begin{aligned} &|a_{m,n}|\{|a_{m-1,n}| + |\mu_{m-1,n}|(1+\alpha)\} \\ &\quad + |\mu_{m,n}|\{|a_{m-1,n}| + |\mu_{m-1,n}|(1+\alpha)\}(1+\alpha) \le R^2. \end{aligned} \tag{4.316}$$

Let $\varphi \in S_\alpha$ be a bounded function defined on Ω and let $\{x_{m,n}\}$ be a solution of system (4.256) with initial condition φ. Then, from (4.256) and (4.315), for all nonnegative integer n, one has

$$\begin{aligned} |x_{1,n}| &= |\mu_{0,n}x_{0,n}(1 - x_{-\sigma,n-\tau}) - a_{0,n}x_{0,n+1}| \\ &\le (|a_{0,n}| + (1+\alpha)|\mu_{0,n}|) \cdot \alpha \le \alpha R \le \alpha. \end{aligned} \tag{4.317}$$

Also, from (4.275) and (4.316),

$$\begin{aligned} |x_{2,n}| &\le \mu_{1,n}(-a_{0,n}x_{0,n+1} + \mu_{0,n}x_{0,n}(1 - x_{-\sigma,n-\tau}))(1 - x_{1-\sigma,n-\tau}) \\ &\quad + |a_{1,n}(-a_{0,n+1}x_{0,n+2} + \mu_{0,n+1}x_{0,n+1}(1 - x_{-\sigma,n+1-\tau}))| \\ &\le \{|\mu_{1,n}|(|a_{0,n}| + |\mu_{0,n}|(1+\alpha))(1+\alpha) \\ &\quad + |a_{1,n}|(|a_{0,n+1}| + |\mu_{0,n+1}|(1+\alpha))\} \cdot \alpha \\ &\le \alpha R^2 \le \alpha \quad \forall n \in N_0. \end{aligned} \tag{4.318}$$

Assume that for some integer $m \ge 2$,

$$|x_{i,j}| \le \alpha R^i \le \alpha \quad \forall 1 \le i \le m \text{ and all } j \in N_0. \tag{4.319}$$

Then, for all $n \ge 0$, from (4.275) and (4.316),

$$\begin{aligned} |x_{m+1,n}| &\le |\mu_{m,n}|(|a_{m-1,n}| + |\mu_{m-1,n}|(1+\alpha))(1+\alpha) \cdot \alpha R^{m-1} \\ &\quad + |a_{m,n}|(|a_{m-1,n+1}| + |\mu_{m-1,n+1}|(1+\alpha)) \cdot \alpha R^{m-1} \\ &\le \alpha R^{m+1} \le \alpha \quad \forall n \in N_0. \end{aligned} \tag{4.320}$$

By induction, $|x_{m,n}| \leq \alpha R^m$ for all $(m,n) \in N_1 \times N_0$, that is, system (4.256) is exponentially asymptotically stable. The proof is complete. □

Corollary 4.58. *Assume that there exists a constant* $r \in (0,1)$ *such that*

$$|a_{m,n}| + |\mu_{m,n}| \leq r \quad \textit{for any } (m,n) \in N_0 \times N_0 = N_0^2. \tag{4.321}$$

Then system (4.256) *is exponentially asymptotically stable.*

In fact, (4.321) implies (4.314). Hence, the proof is complete.

Example 4.59. Consider the delayed 2D discrete logistic system

$$x_{m+1,n} + a_{m,n}x_{m,n+1} = \mu_{m,n}x_{m,n}(1 - x_{m-2,n-1}), \quad (m,n) \in N_0^2, \tag{4.322}$$

where

$$\begin{aligned}
a_{0,n} &= \frac{1}{2}\left(\frac{7}{8}\right)^{n+1}, & \mu_{0,n} &= \frac{1}{4}\left(\frac{7}{8}\right)^{n+1} \quad \forall n \in N_0, \\
a_{m,n} &= \frac{(-1)^{m+n}}{2}, & \mu_{m,n} &= \frac{7}{32} \quad \forall (m,n) \in D_1 \cup D_2 \cup D_3, \\
a_{m,n} &= \frac{3}{7}, & \mu_{m,n} &= \frac{(-1)^{m+n}}{2(1+(7/8)^{m+n-3})} \quad \forall (m,n) \in D_4,
\end{aligned} \tag{4.323}$$

in which D_1, D_2, D_3, and D_4 are defined in Theorem 4.53.

Obviously, $\sigma = 2$ and $\tau = 1$. Let $\xi = 7/8$ and $M = 1$. Then, it is easy to see that all the conditions of Theorem 4.53 hold. Hence, system (4.322) is D-B-exponentially stable.

Example 4.60. Consider the delayed 2D discrete logistic system

$$x_{m+1,n} + a_{m,n}x_{m,n+1} = \mu_{m,n}x_{m,n}(1 - x_{m-1,n-2}), \quad (m,n) \in N_0^2, \tag{4.324}$$

where

$$a_{m,n} = 1 - \frac{2}{m+2}, \quad \mu_{m,n} = \frac{1}{m+2} \quad \forall (m,n) \in N_0^2. \tag{4.325}$$

Clearly, there exist two constants, $\alpha = 1$ and $C = 1$, and an integer $k = 1$, such that all the conditions of Theorem 4.51 hold. Hence, system (4.324) is stable.

4.4.2. Stability of generalized 2D discrete systems

Consider the delayed generalized 2D discrete systems of the form

$$x_{m+1,n} = f(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau}), \tag{4.326}$$

where σ and τ are positive integers, m and n are nonnegative integers, and $f : Z^2 \times R^3 \to R$ is a real function containing the logistic map as a special case. Obviously, if

$$f(m,n,x,y,z) = \mu_{m,n}x(1-x) - a_{m,n}y, \tag{4.327}$$

or

$$f(m,n,x,y,z) = \mu_{m,n}x(1-z) - a_{m,n}y, \tag{4.328}$$

or

$$f(m,n,x,y,z) = 1 - \mu x^2 - ay, \tag{4.329}$$

or

$$f(m,n,x,y,z) = b_{m,n}x - a_{m,n}y - p_{m,n}z, \tag{4.330}$$

then system (4.326) becomes, respectively,

$$x_{m+1,n} + a_{m,n}x_{m,n+1} = \mu_{m,n}x_{m,n}\left(1 - x_{m,n}\right), \tag{4.331}$$

or

$$x_{m+1,n} + a_{m,n}x_{m,n+1} = \mu_{m,n}x_{m,n}\left(1 - x_{m-\sigma,n-\tau}\right), \tag{4.332}$$

or

$$x_{m+1,n} + a_{m,n}x_{m,n+1} = 1 - \mu\left(x_{m,n}\right)^2, \tag{4.333}$$

or

$$x_{m+1,n} + a_{m,n}x_{m,n+1} - b_{m,n}x_{m,n} + p_{m,n}x_{m-\sigma,n-\tau} = 0. \tag{4.334}$$

Systems (4.331), (4.332), and (4.333) are regular 2D discrete logistic systems of different forms, and particulary system (4.334) has been studied in the literature.

If $a_{m,n} = 0$, $\mu_{m,n} = \mu$, and $n = n_0$ is fixed, then system (4.331) becomes the 1D logistic system

$$x_{m+1,n_0} = \mu x_{m,n_0}\left(1 - x_{m,n_0}\right), \tag{4.335}$$

where μ is a parameter. System (4.335) has been intensively investigated in the literature. Hence, system (4.326) is quite general.

This section is concerned with the stability of solutions of system (4.326), in which some sufficient conditions for the stability and exponential stability of system (4.326) will be derived.

Let $\Omega = N_{-\sigma} \times N_{-\tau} \setminus N_1 \times N_0$. It is obvious that for any given function $\varphi = \{\varphi_{m,n}\}$ defined on Ω, it is easy to construct by induction a double sequence $\{x_{m,n}\}$ that equals initial conditions φ on Ω and satisfies (4.326) on $N_1 \times N_0$.

Definition 4.61. Let $x^* \in R$ be a constant. If x^* is a root of the equation

$$x - f(m,n,x,x,x) = 0 \quad \text{for any } (m,n) \in N_0^2, \tag{4.336}$$

then x^* is said to be a fixed point or equilibrium point of system (4.326). The set of all fixed points of system (4.326) is called a fixed plane or equilibrium plane of the system.

It is easy to see that $x^* = 0$ is a fixed point of systems (4.331), (4.332), and (4.334), and $x^* = (-(a+1) \pm \sqrt{(a+1)^2 + 4\mu})(2\mu)^{-1}$ are two fixed points of system (4.333).

Let x^* be a fixed point of system (4.326) and let $\varphi = \{\varphi_{m,n}\}$ be a function defined on Ω, and let

$$\|\varphi\|_{x^*} = \sup\{|\varphi_{m,n} - x^*| \mid (m,n) \in \Omega\}. \tag{4.337}$$

For any positive number $\delta > 0$, let $S_\delta(x^*) = \{\varphi \mid\mid \|\varphi\|_{x^*} < \delta\}$.

Definition 4.62. Let $x^* \in R$ be a fixed point of system (4.326). If for any $\varepsilon > 0$, there exists a positive constant $\delta > 0$ such that for any given bounded function $\varphi = \{\varphi_{m,n}\}$ defined on $\Omega, \varphi \in S_\delta(x^*)$ implies that the solution $x = \{x_{m,n}\}$ of system (4.326) with the initial condition φ satisfies

$$|x_{m,n} - x^*| < \varepsilon \quad \forall (m,n) \in N_1 \times N_0, \tag{4.338}$$

then system (4.326) is said to be stable about the fixed point x^*.

Definition 4.63. Let $x^* \in R$ be a fixed point of system (4.326). If there exist positive constants $M > 0$ and $\xi \in (0,1)$ such that for any given constant $\delta \in (0,M)$ and any given bounded function $\varphi = \{\varphi_{m,n}\}$ defined on $\Omega, \varphi \in S_\delta(x^*)$ which implies that the solution $\{x_{m,n}\}$ of system (4.326) with the initial condition φ satisfies

$$|x_{m,n} - x^*| < M\xi^{m+n}, \quad (m,n) \in N_1 \times N_0, \tag{4.339}$$

then system (4.326) is said to be double-variable-bounded-initially exponentially stable, or D-B-exponentially stable, about the fixed point x^*.

Definition 4.64. Let $x^* \in R$ be a fixed point of system (4.326). If there exist positive constants $M > 0$ and $\xi \in (0,1)$ such that for any given bounded number $\delta \in (0,M)$ and any given bounded function $\varphi = \{\varphi_{m,n}\}$ defined on $\Omega, \varphi \in S_\delta(x^*)$

which implies that the solution $\{x_{m,n}\}$ of system (4.326) with the initial condition φ satisfies

$$|x_{m,n} - x^*| < M\xi^m, \quad (m,n) \in N_1 \times N_0, \tag{4.340}$$

then system (4.326) is said to be exponentially asymptotically stable about the fixed point x^*.

Obviously, if system (4.326) is D-B-exponentially stable or exponentially asymptotically stable, then it is stable.

Definition 4.65. Let $f(m,n,x,y,z)$ be a function defined on $Z^2 \times D$ and let $(x_0,y_0,z_0) \in D$ be a fixed inner point, where $D \subset R^3$. If, for any positive constant $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any $|x-x_0| < \delta$, $|y-y_0| < \delta$, $|z-z_0| < \delta$,

$$|f(m,n,x,y,z) - f(m,n,x_0,y_0,z_0)| < \varepsilon \quad \text{for any } (m,n) \in N_0^2, \tag{4.341}$$

then $f(m,n,x,y,z)$ is said to be uniformly continuous at the point (x_0,y_0,z_0) (over m and n). If the partial derivative functions $f_x'(m,n,x,y,z)$, $f_y'(m,n,x,y,z)$, and $f_z'(m,n,x,y,z)$ are all uniformly continuous at (x_0,y_0,z_0), then $f(m,n,x,y,z)$ is said to be uniformly differentiable at (x_0,y_0,z_0).

Let D be an open subset of R^3. If $f(m,n,x,y,z)$ is uniformly continuous at any point $(x,y,z) \in D$, then it is said to be uniformly continuous on D.

Obviously, if $f(m,n,x,y,z)$ and $g(m,n,x,y,z)$ are uniformly continuous at (x,y,z), then

$$af(m,n,x,y,z),\ |f(m,n,x,y,z)|,\ f(m,n,x,y,z) + g(m,n,x,y,z) \tag{4.342}$$

are also uniformly continuous at (x,y,z) for any constant $\alpha \in R$.

Lemma 4.66. *Let $D \subset R^3$ be an open convex domain and $(x_0,y_0,z_0) \in D$. Assume that the function $f(m,n,x,y,z)$ is continuously differentiable on D for any fixed m and n. Then for any $(\tilde{x},\tilde{y},\tilde{z}) \in D$ and any $(m,n) \in N_0^2$, there exists a constant $t_0 = t(m,n,\tilde{x},\tilde{y},\tilde{z}) \in (0,1)$ such that*

$$\begin{aligned} f(m,n,\tilde{x},\tilde{y},\tilde{z}) &- f(m,n,x_0,y_0,z_0) \\ &= f_x'(m,n,x_0+t_0(\tilde{x}-x_0), y_0+t_0(\tilde{y}-y_0), z_0+t_0(\tilde{z}-z_0))(\tilde{x}-x_0) \\ &\quad + f_y'(m,n,x_0+t_0(\tilde{x}-x_0), y_0+t_0(\tilde{y}-y_0), z_0+t_0(\tilde{z}-z_0))(\tilde{y}-y_0) \\ &\quad + f_z'(m,n,x_0+t_0(\tilde{x}-x_0), y_0+t_0(\tilde{y}-y_0), z_0+t_0(\tilde{z}-z_0))(\tilde{z}-z_0). \end{aligned} \tag{4.343}$$

Proof. Let $g(t) = f(m,n,x_0+t(\tilde{x}-x_0), y_0+t(\tilde{y}-y_0), z_0+t(\tilde{z}-z_0))$. Then, from the given conditions, the function $g(t)$ is continuous differentiable on [0,1]. Hence, from the mean value theorem, there exists a constant $t_0 \in (0,1)$ such that $g(1) - g(0) = g'(t_0)$, that is, Lemma 4.66 holds. The proof is completed. □

Theorem 4.67. *Assume that x^* is a fixed point of system* (4.326), *the function $f(m,n,x,y,z)$ is both continuously differentiable on R^3 for any fixed $(m,n) \in N_0^2$ and uniformly continuously differentiable at the point $(x^*,x^*,x^*) \in R^3$, and there exists a constant $r \in (0,1)$ such that for any $(m,n) \in N_0^2$,*

$$\left| f_x'(m,n,x^*,x^*,x^*) \right| + \left| f_y'(m,n,x^*,x^*,x^*) \right| + \left| f_z'(m,n,x^*,x^*,x^*) \right| \le r. \tag{4.344}$$

Then system (4.326) *is stable.*

Proof. Since the function $f(m,n,x,y,z)$ is uniformly continuously differentiable at the point (x^*,x^*,x^*), there exists a positive number $M > 0$ such that for any $(m,n) \in N_0^2$ and any $(x,y,z) \in R^3$ satisfying $|x - x^*| < M, |y - x^*| < M$ and $|z - x^*| < M$,

$$\left| f_x'(m,n,x,y,z) \right| + \left| f_y'(m,n,x,y,z) \right| + \left| f_z'(m,n,x,y,z) \right| \le 1. \tag{4.345}$$

In view of the given conditions and Lemma 4.66, for any $m \ge 0$ and $n \ge 0$, and any point $(x,y,z) \in R^3$ which satisfies $|x - x^*| < M$, $|y - x^*| < M$ and $|z - x^*| < M$, there exists a constant $t_0 = t(m,n,x,y,z) \in (0,1)$ such that

$$\begin{aligned} f(m,n,x,y,z) &- f(m,n,x^*,x^*,x^*) \\ &= f_x'(m,n,\lambda,\eta,\theta)(x - x^*) + f_y'(m,n,\lambda,\eta,\theta)(y - x^*) \\ &\quad + f_z'(m,n,\lambda,\eta,\theta)(z - x^*), \end{aligned} \tag{4.346}$$

where $\lambda = x^* + t_0(x - x^*)$, $\eta = x^* + t_0(y - x^*)$, and $\theta = x^* + t_0(z - x^*)$. Obviously,

$$\left| \lambda - x^* \right| \le \left| x - x^* \right|, \qquad \left| \eta - x^* \right| \le \left| y - x^* \right|, \qquad \left| \theta - x^* \right| \le \left| z - x^* \right|. \tag{4.347}$$

For any sufficiently small number $\varepsilon > 0$, without loss of generality, let $\varepsilon < M$ and $\delta = \varepsilon$, and let $\varphi = \{\varphi_{m,n}\}$ be a given bounded function defined on Ω which satisfies $|\varphi_{m,n} - x^*| < \delta$ for all $(m,n) \in \Omega$. Let the sequence $\{x_{m,n}\}$ be a solution of system (4.326) with the initial condition φ. In view of (4.326) and the following inequalities:

$$\left| x_{0,0} - x^* \right| \le \delta < M, \qquad \left| x_{0,1} - x^* \right| \le \delta < M, \qquad \left| x_{-\sigma,-\tau} - x^* \right| \le \delta < M, \tag{4.348}$$

it follows from (4.345), (4.346), and Lemma 4.66 that there exists a constant

$$t_0 = t(0,0,x_{0,0},x_{0,1},x_{-\sigma,-\tau}) \in (0,1), \tag{4.349}$$

such that

$$\begin{aligned} |x_{1,0}-x^*| &= |f(0,0,x_{0,0},x_{0,1},x_{-\sigma,-\tau})-f(0,0,x^*,x^*,x^*)| \\ &\le |f_x'(0,0,\lambda,\eta,\theta)|\,|x_{0,0}-x^*| + |f_y'(0,0,\lambda,\eta,\theta)|\,|x_{0,1}-x^*| \\ &\quad + |f_z'(0,0,\lambda,\eta,\theta)|\,|x_{-\sigma,-\tau}-x^*| \le \delta \le \varepsilon < M, \end{aligned} \tag{4.350}$$

where $\lambda = x^* + t_0(x_{0,0}-x^*)$, $\eta = x^* + t_0(x_{0,1}-x^*)$, and $\theta = x^* + t_0(x_{-\sigma,-\tau}-x^*)$. Similarly, from (4.326), (4.345), and (4.346), one has

$$|x_{1,1}-x^*| = |f(0,1,x_{0,1},x_{0,2},x_{-\sigma,1-\tau})-f(0,1,x^*,x^*,x^*| \le \varepsilon < M. \tag{4.351}$$

In general, for any integer $n \ge 0$, $|x_{1,n}-x^*| \le \varepsilon < M$.

Assume that for a certain integer $k \ge 1$,

$$|x_{i,n}-x^*| \le \varepsilon < M \quad \text{for any } i \in \{1,2,\dots,k\},\ n \ge 0. \tag{4.352}$$

Then, it follows from (4.326), (4.345), and (4.346) that there exists a constant

$$t_0 = t(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau}) \in (0,1), \tag{4.353}$$

such that

$$\begin{aligned} |x_{k+1,n}-x^*| &= |f(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau})-f(k,n,x^*,x^*,x^*)| \\ &\le |f_x'(k,n,\lambda,\eta,\theta)|\,|x_{k,n}-x^*| + |f_y'(k,n,\lambda,\eta,\theta)|\,|x_{k,n+1}-x^*| \\ &\quad + |f_z'(k,n,\lambda,\eta,\theta)|\,|x_{k-\sigma,n-\tau}-x^*| \\ &\le \big(|f_x'(k,n,\lambda,\eta,\theta)| + |f_y'(k,n,\lambda,\eta,\theta)| + |f_z'(k,n,\lambda,\eta,\theta)|\big)\cdot\varepsilon \\ &\le \varepsilon, \end{aligned} \tag{4.354}$$

where $\lambda = x^*+t_0(x_{k,n}-x^*)$, $\eta = x^*+t_0(x_{k,n+1}-x^*)$, and $\theta = x^*+t_0(x_{k-\sigma,n-\tau}-x^*)$. Hence, by induction, $|x_{m,n}-x^*| \le \varepsilon$ for any $(m,n) \in N_1 \times N_0$, that is, system (4.326) is stable. The proof is completed. □

Similar to the above proof of Theorem 4.67, it is easy to obtain the following result.

Theorem 4.68. *Assume that x^* is a fixed point of system* (4.326), *and the function $f(m,n,x,y,z)$ is continuously differentiable on R^3 for any fixed m and n. Further,* *assume that there exists an open subset $D \subset R^3$ such that $(x^*, x^*, x^*) \in D$, and for any $(m, n) \in N_0^2$ and any $(x, y, z) \in D$,*

$$|f'_x(m,n,x,y,z)| + |f'_y(m,n,x,y,z)| + |f'_z(m,n,x,y,z)| \leq 1. \tag{4.355}$$

Then system (4.326) *is stable.*

From Theorems 4.67 and 4.68, one obtains the following results.

Corollary 4.69. *Assume that there exists a constant $r \in (0,1)$ such that*

$$|\mu_{m,n}| + |a_{m,n}| \leq r \quad \forall m \geq 0, \ n \geq 0. \tag{4.356}$$

Then system (4.331) *and* (4.332) *are both stable.*

In fact, system (4.331) is a special case of system (4.326) when

$$f(m,n,x,y,z) \equiv \mu_{m,n}x(1-x) - a_{m,n}y. \tag{4.357}$$

In view of (4.356), it is obvious that the function $f(m,n,x,y,z)$ is both continuously differentiable on R^3 for any fixed $(m,n) \in N_0^2$ and uniformly continuously differentiable at the point $(0,0,0)$. Since $x^* = 0$ is a fixed point of system (4.331) and (4.332), one has

$$\begin{aligned} f'_x(m,n,x^*,x^*,x^*) &= \mu_{m,n}, \\ f'_y(m,n,x^*,x^*,x^*) &= -a_{m,n}, \\ f'_z(m,n,x^*,x^*,x^*) &= 0. \end{aligned} \tag{4.358}$$

Hence, (4.356) implies (4.344). By Theorem 4.67, system (4.331) and (4.332) are both stable.

Corollary 4.70. *System* (4.333) *has fixed points*

$$x^* = \left(-(a+1) \pm \sqrt{(a+1)^2 + 4\mu}\right)(2\mu)^{-1}. \tag{4.359}$$

Assume that there exists a constant $r \in (0,1)$ such that

$$2|\mu \cdot x^*| + |a| \leq r. \tag{4.360}$$

Then system (4.333) *is stable about x^*.*

Corollary 4.71. *Assume that*

$$|a_{m,n}| + |b_{m,n}| + |p_{m,n}| \leq 1 \quad \forall m \geq 0, \ n \geq 0. \tag{4.361}$$

Then system (4.334) *is stable.*

Define four subsets of $N_0 \times N_0$ as follows:

$$B_1 = \{(i,j) : 0 \le i \le \sigma,\ 0 \le j < \tau\}, \quad B_2 = \{(i,j) : 0 \le i \le \sigma,\ j \ge \tau\}$$
$$B_3 = \{(i,j) : i > \sigma,\ 0 \le j < \tau\}, \quad B_4 = \{(i,j) : i > \sigma,\ j \ge \tau\}. \tag{4.362}$$

Obviously, B_1 is a finite set, B_2, B_3, and B_4 are infinite sets, B_1, B_2, B_3, and B_4 are disjoint of one another, and $N_0^2 = B_1 \cup B_2 \cup B_3 \cup B_4$.

Theorem 4.72. *Assume that x^* is a fixed point of system* (4.333), *the function $f(m,n,x,y,z)$ is both continuously differentiable on R^3 for any $(m,n) \in N_0^2$ and uniformly continuously differentiable at the point $(x^*,x^*,x^*) \in R^3$, and there exists a constant $r \in (0,1)$ such that for any $(m,n) \in B_1 \cup B_2$,*

$$|f_x'(m,n,x^*,x^*,x^*)| + |f_y'(m,n,x^*,x^*,x^*)| + r^{-m}|f_z'(m,n,x^*,x^*,x^*)| \le r, \tag{4.363}$$

and for any $(m,n) \in B_3 \cup B_4$,

$$|f_x'(m,n,x^*,x^*,x^*)| + |f_y'(m,n,x^*,x^*,x^*)| + r^{-\sigma}|f_z'(m,n,x^*,x^*,x^*)| \le r. \tag{4.364}$$

Then system (4.326) *is exponentially asymptotically stable.*

Proof. From the given conditions, there exist two positive constants, $M > 0$ and $\xi \in (r,1)$, such that (4.346) holds and for any $(m,n) \in B_1 \cup B_2$,

$$|f_x'(m,n,x^*,x^*,x^*)| + |f_y'(m,n,x^*,x^*,x^*)| + \xi^{-m}|f_z'(m,n,x^*,x^*,x^*)| \le \xi, \tag{4.365}$$

and for any $(m,n) \in B_3 \cup B_4$,

$$|f_x'(m,n,x^*,x^*,x^*)| + |f_y'(m,n,x^*,x^*,x^*)| + \xi^{-\sigma}|f_z'(m,n,x^*,x^*,x^*)| \le \xi \tag{4.366}$$

for $|x-x^*| < M$, $|y-x^*| < M$, and $|z-x^*| < M$.

Let $\delta \in (0,M)$ be a given constant and let $\varphi = \{\varphi_{m,n}\}$ be a given bounded function defined on Ω which satisfies $|\varphi_{m,n} - x^*| < \delta$ for all $(m,n) \in \Omega$. Let the sequence $\{x_{m,n}\}$ be a solution of system (4.326) with the initial condition φ. In view of (4.326) and the following inequalities:

$$|x_{0,0} - x^*| \le \delta < M, \qquad |x_{0,1} - x^*| \le \delta < M, \qquad |x_{-\sigma,-\tau}| \le \delta < M, \tag{4.367}$$

it follows from (4.346), (4.366), and Lemma 4.66 that there exists a constant

$$t_0 = t(0,0,x_{0,0},x_{0,1},x_{-\sigma,-\tau}) \in (0,1), \tag{4.368}$$

such that

$$\begin{aligned} |x_{1,0} - x^*| &= |f(0,0,x_{0,0},x_{0,1},x_{-\sigma,-\tau}) - f(0,0,x^*,x^*,x^*)| \\ &\le |f'_x(0,0,\lambda,\eta,\theta)|\,|x_{0,0} - x^*| + |f'_y(0,0,\lambda,\eta,\theta)|\,|x_{0,1} - x^*| \\ &\quad + |f'_z(0,0,\lambda,\eta,\theta)|\,|x_{-\sigma,-\tau} - x^*| \le M\xi, \end{aligned} \tag{4.369}$$

where $\lambda = x^* + t_0(x_{0,0} - x^*)$, $\eta = x^* + t_0(x_{0,1} - x^*)$, and $\theta = x^* + t_0(x_{-\sigma,-\tau} - x^*)$. Similarly, from (4.326), (4.346), and (4.366), one has

$$|x_{1,1} - x^*| = |f(0,1,x_{0,1},x_{0,2},x_{-\sigma,1-\tau}) - f(0,1,x^*,x^*,x^*)| \le M\xi. \tag{4.370}$$

In general, for any integer $n \ge 0$, $|x_{1,n} - x^*| \le M\xi$.

Assume that for a certain integer $k \in \{1,\dots,\sigma\}$,

$$|x_{i,n} - x^*| \le M\xi^i \quad \text{for any } i \in \{1,2,\dots,k\},\ n \ge 0. \tag{4.371}$$

Then, $(k,n) \in B_1 \cup B_2$ and $(k-\sigma, n-\tau) \in \Omega$. From (4.346), (4.365), and Lemma 4.66, there exists a constant $t_0 = t(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau}) \in (0,1)$ such that

$$\begin{aligned} |x_{k+1,n} - x^*| &= |f(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau}) - f(k,n,x^*,x^*,x^*)| \\ &\le |f'_x(k,n,\lambda,\eta,\theta)|\,|x_{k,n} - x^*| + |f'_y(k,n,\lambda,\eta,\theta)|\,|x_{k,n+1} - x^*| \\ &\quad + |f'_z(k,n,\lambda,\eta,\theta)|\,|x_{k-\sigma,n-\tau} - x^*| \\ &\le M\xi^k |f'_x(k,n,\lambda,\eta,\theta)| + M\xi^k |f'_y(k,n,\lambda,\eta,\theta)| \\ &\quad + M\xi^k |f'_z(k,n,\lambda,\eta,\theta)| \le M\xi^{k+1}, \end{aligned} \tag{4.372}$$

where $\lambda = x^* + t_0(x_{k,n} - x^*)$, $\eta = x^* + t_0(x_{k,n+1} - x^*)$, and $\theta = x^* + t_0(x_{k-\sigma,n-\tau} - x^*)$. Hence, by induction, $|x_{m,n} - x^*| \le M\xi^m$ for any $(m,n) \in \{1,2,\dots,\sigma+1\}$ and $n \ge 0$.

Assume that for a certain integer $k \ge \sigma + 1$,

$$|x_{i,n} - x^*| \le M\xi^i \quad \text{for any } i \in \{1,2,\dots,k\},\ n \ge 0. \tag{4.373}$$

Then, $(k,n) \in B_3 \cup B_4$, $(k-\sigma, n-\tau) \notin \Omega$. Hence, from (4.326), (4.346), (4.366), and Lemma 4.66, there exists a constant $t_0 = t(k,n,x_{k,n},x_{k,n+1},x_{k-\sigma,n-\tau}) \in (0,1)$ such that

$$\begin{aligned} |x_{k+1,n} - x^*| &\le |f'_x(k,n,\lambda,\eta,\theta)| \cdot M\xi^k + |f'_y(k,n,\lambda,\eta,\theta)| \cdot M\xi^k \\ &\quad + |f'_z(k,n,\lambda,\eta,\theta)| \cdot M\xi^{k-\sigma} \\ &= M\xi^k(|f'_x(k,n,\lambda,\eta,\theta)| + |f'_y(k,n,\lambda,\eta,\theta)| \\ &\quad + \xi^{-\sigma}|f'_z(k,n,\lambda,\eta,\theta)|) \le M\xi^{k+1}, \end{aligned} \tag{4.374}$$

where $\lambda = x^* + t_0(x_{k,n} - x^*)$, $\eta = x^* + t_0(x_{k,n+1} - x^*)$, and $\theta = x^* + t_0(x_{k-\sigma,n-\tau} - x^*)$. By induction, $|x_{m,n} - x^*| \le M\xi^m$ for any $(m,n) \in N_1 \times N_0$, that is, system (4.326) is exponentially asymptotically stable. The proof is completed. □

From Theorem 4.72, it is easy to obtain the following corollaries.

Corollary 4.73. *Assume that there exists a constant $r \in (0,1)$ such that*

$$|\mu_{m,n}| + |a_{m,n}| \le r \quad \forall m \ge 0,\ n \ge 0. \tag{4.375}$$

Then systems (4.331) and (4.332) are both exponentially asymptotically stable.

Corollary 4.74. *System (4.333) has fixed points*

$$x^* = \left(-(a+1) \pm \sqrt{(a+1)^2 + 4\mu}\right)(2\mu)^{-1}. \tag{4.376}$$

Assume that there exists a constant $r \in (0,1)$ such that

$$2|\mu \cdot x^*| + |a| \le r. \tag{4.377}$$

Then system (4.333) is exponentially asymptotically stable.

Corollary 4.75. *Assume that there exists a constant $r \in (0,1)$ such that for $(m,n) \in B_1 \cup B_2$,*

$$|a_{m,n}| + |b_{m,n}| + r^{-m}|p_{m,n}| \le r, \tag{4.378}$$

and for $(m,n) \in B_3 \cup B_4$,

$$|a_{m,n}| + |b_{m,n}| + r^{-\sigma}|p_{m,n}| \le r. \tag{4.379}$$

Then system (4.334) *is exponentially asymptotically stable.*

Let

$$\begin{aligned} D_1 &= \{(m,n) : 1 \le m \le \sigma,\ 0 \le n < \tau\}, \quad & D_2 &= \{(m,n) : m > \sigma,\ 0 \le n < \tau\}, \\ D_3 &= \{(m,n) : 1 \le m \le \sigma,\ n \ge \tau\}, \quad & D_4 &= \{(m,n) : m > \sigma,\ n \ge \tau\}. \end{aligned} \tag{4.380}$$

Obviously, D_1, D_2, D_3, D_4 are disjoint of one another, and $N_1 \times N_0 = D_1 \cup D_2 \cup D_3 \cup D_4$.

Theorem 4.76. *Assume that* x^* *is a fixed point of system* (4.326), $f(m,n,x,y,z)$ *is both continuously differentiable on* R^3 *for any* $(m,n) \in N_0^2$ *and uniformly continuously differentiable at the point* $(x^*,x^*,x^*) \in R^3$. *Further, assume that there exist a constant* $r \in (0,1)$ *and an open subset* $D \subset R^3$ *with* $(x^*,x^*,x^*) \in D$ *such that for any* $(x,y,z) \in D$ *and any* $n \ge 0$,

$$|f_x'(0,n,x,y,z)| + |f_y'(0,n,x,y,z)| + |f_z'(0,n,x,y,z)| \le r^{n+1}, \tag{4.381}$$

and for all $(m,n) \in D_1 \cup D_3$,

$$\begin{aligned} &|f_x'(m,n,x^*,x^*,x^*)| \\ &\quad + r|f_y'(m,n,x^*,x^*,x^*)| + r^{-m-n}|f_z'(m,n,x^*,x^*,x^*)| \le r, \end{aligned} \tag{4.382}$$

and for all $(m,n) \in D_2 \cup D_4$,

$$\begin{aligned} &|f_x'(m,n,x^*,x^*,x^*)| \\ &\quad + r|f_y'(m,n,x^*,x^*,x^*)| + r^{-\sigma-\tau}|f_z'(m,n,x^*,x^*,x^*)| \le r. \end{aligned} \tag{4.383}$$

Then system (4.326) *is D-B-exponentially stable.*

Proof. From the given conditions, and from (4.381), (4.382), and (4.383), there exist positive constants $M > 0$ and $\xi \in (r,1)$, such that (4.346) holds and for all $n \geq 0$,

$$\left| f_x'(0,n,x,y,z) \right| + \left| f_y'(0,n,x,y,z) \right| + \left| f_z'(0,n,x,y,z) \right| \leq \xi^{n+1}, \tag{4.384}$$

and for all $(m,n) \in D_1 \cup D_3$,

$$\begin{aligned} &\left| f_x'(m,n,x^*,x^*,x^*) \right| \\ &\qquad + \xi \left| f_y'(m,n,x^*,x^*,x^*) \right| + \xi^{-m-n} \left| f_z'(m,n,x^*,x^*,x^*) \right| \leq \xi, \end{aligned} \tag{4.385}$$

and for all $(m,n) \in D_2 \cup D_4$,

$$\begin{aligned} &\left| f_x'(m,n,x^*,x^*,x^*) \right| \\ &\qquad + \xi \left| f_y'(m,n,x^*,x^*,x^*) \right| + \xi^{-\sigma-\tau} \left| f_z'(m,n,x^*,x^*,x^*) \right| \leq \xi \end{aligned} \tag{4.386}$$

for $|x - x^*| < M$, $|y - x^*| < M$ and $|z - x^*| < M$.

Let $\delta \in (0,M)$ be a constant and let $\varphi = \{\varphi_{m,n}\}$ be a given bounded function defined on Ω which satisfies $|\varphi_{m,n} - x^*| < \delta$ for all $(m,n) \in \Omega$. Let the sequence $\{x_{m,n}\}$ be a solution of system (4.326) with the initial condition φ. In view of (4.326) and the following inequalities:

$$\left| x_{0,n} - x^* \right| \leq \delta < M, \qquad \left| x_{0,n+1} - x^* \right| \leq \delta < M, \qquad \left| x_{-\sigma,n-\tau} \right| \leq \delta < M, \tag{4.387}$$

it follows from (4.346) and (4.384) that there exists a constant

$$t_{0,n} = t\left(0,n,x_{0,n},x_{0,n+1},x_{-\sigma,n-\tau}\right) \in (0,1) \tag{4.388}$$

such that for any $n \in N_0$,

$$\begin{aligned} \left| x_{1,n} - x^* \right| &= \left| f\left(0,n,x_{0,n},x_{0,n+1},x_{-\sigma,n-\tau}\right) - f\left(0,n,x^*,x^*,x^*\right) \right| \\ &\leq \left| f_x'\left(0,n,\lambda_{0,n},\eta_{0,n},\theta_{0,n}\right) \right| \left| x_{0,n} - x^* \right| \\ &\quad + \left| f_y'\left(0,n,\lambda_{0,n},\eta_{0,n},\theta_{0,n}\right) \right| \left| x_{0,n+1} - x^* \right| \\ &\quad + \left| f_z'\left(0,n,\lambda_{0,n},\eta_{0,n},\theta_{0,n}\right) \right| \left| x_{-\sigma,n-\tau} - x^* \right| \leq M\xi^{n+1}, \end{aligned} \tag{4.389}$$

where $\lambda_{0,n} = x^* + t_{0,n}(x_{0,n} - x^*)$, $\eta_{0,n} = x^* + t_{0,n}(x_{0,n+1} - x^*)$, and $\theta_{0,n} = x^* + t_{0,n}(x_{-\sigma,n-\tau} - x^*)$.

Assume that for some $m \in \{1,\dots,\sigma\}$,

$$|x_{i,j} - x^*| \le M\xi^{i+j} \quad \forall 1 \le i \le m,\ j \in N_0. \tag{4.390}$$

Then, for all $n \ge 0$, one has $(m-\sigma, n-\tau) \in \Omega$ and $(m,n) \in D_1 \cup D_3$. Hence, it follows from (4.346) and (4.385) that there exists a constant $t_{m,n} = t(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau}) \in (0,1)$ such that

$$\begin{aligned}
|x_{m+1,n} - x^*| &= |f(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau}) - f(m,n,x^*,x^*,x^*)| \\
&\le |f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m,n} - x^*| \\
&\quad + |f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m,n+1} - x^*| \\
&\quad + |f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m-\sigma,n-\tau} - x^*| \\
&\le |f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \cdot M\xi^{m+n} \\
&\quad + |f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \cdot M\xi^{m+n+1} \\
&\quad + |f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \cdot M \le M\xi^{m+n+1},
\end{aligned} \tag{4.391}$$

where $\lambda_{m,n} = x^* + t_{m,n}(x_{m,n} - x^*)$, $\eta_{m,n} = x^* + t_{m,n}(x_{m,n+1} - x^*)$, and $\theta_{m,n} = x^* + t_{m,n}(x_{m-\sigma,n-\tau} - x^*)$. By induction, $|x_{m,n} - x^*| \le M\xi^{m+n}$ for all $(m,n) \in \{1,2,\dots,\sigma+1\}$ and all $n \ge 0$.

Assume that for some $m \ge \sigma + 1$,

$$|x_{i,n} - x^*| \le M\xi^{i+n} \quad \forall 1 \le i \le m, \text{ and all } n \in N_0. \tag{4.392}$$

Then, for all $n \ge 0$, one has $(m-\sigma, n-\tau) \notin \Omega$ and $(m,n) \in D_2 \cup D_4$. Hence, it follows from (4.346) and (4.386) that there exists a constant

$$t_{m,n} = t(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau}) \in (0,1) \tag{4.393}$$

such that

$$\begin{aligned}
|x_{m+1,n} - x^*| &= |f(m,n,x_{m,n},x_{m,n+1},x_{m-\sigma,n-\tau}) - f(m,n,x^*,x^*,x^*)| \\
&\le |f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m,n} - x^*| \\
&\quad + |f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m,n+1} - x^*| \\
&\quad + |f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\,|x_{m-\sigma,n-\tau} - x^*| \\
&\le |f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \cdot M\xi^{m+n} \\
&\quad + |f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \cdot M\xi^{m+n+1} \\
&\quad + |f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \cdot M\xi^{m+n-\sigma-\tau} \\
&= M\xi^{m+n}\big(|f'_x(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \\
&\quad + \xi|f'_y(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})| \\
&\quad + \xi^{-\sigma-\tau}|f'_z(m,n,\lambda_{m,n},\eta_{m,n},\theta_{m,n})|\big) \\
&\le M\xi^{m+n+1},
\end{aligned} \tag{4.394}$$

where $\lambda_{m,n} = x^* + t_{m,n}(x_{m,n} - x^*)$, $\eta_{m,n} = x^* + t_{m,n}(x_{m,n+1} - x^*)$, and $\theta_{m,n} = x^* + t_{m,n}(x_{m-\sigma,n-\tau} - x^*)$. By induction, $|x_{m,n} - x^*| \le M\xi^{m+n}$ for any $(m,n) \in N_1 \times N_0$, that is, system (4.326) is D-B-exponentially stable. The proof is completed. □

From Theorem 4.76, it is easy to obtain the following corollaries.

Corollary 4.77. *Assume that there exist two constants $r \in (0,1)$ and $C \in (0,1)$ such that*

$$\begin{aligned}
|\mu_{0,n}| + |a_{0,n}| &\le Cr^{n+1} \quad \forall n \ge 0, \\
|\mu_{m,n}| + r|a_{m,n}| &\le r \quad \text{for any } (m,n) \in N_1 \times N_0.
\end{aligned} \tag{4.395}$$

Then, system (4.331) *and* (4.332) *are both D-B-exponentially stable.*

Corollary 4.78. *Assume that there exists a constant $r \in (0,1)$ such that*

$$\begin{aligned}
|a_{0,n}| + |b_{0,n}| + |p_{0,n}| &\le r^{n+1} \quad \text{for any } n \ge 0, \\
r|a_{m,n}| + |b_{m,n}| + r^{-m-n}|p_{m,n}| &\le r \quad \text{for any } (m,n) \in D_1 \cup D_3, \\
r|a_{m,n}| + |b_{m,n}| + r^{-\sigma-\tau}|p_{m,n}| &\le r \quad \text{for any } (m,n) \in D_2 \cup D_4.
\end{aligned} \tag{4.396}$$

Then, system (4.334) *is D-B-exponentially stable.*

4.5. L_2 stability in parabolic Volterra difference equations

Consider the linear parabolic Volterra difference equation

$$\Delta_2\left(u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j}\right) + \sum_{i=1}^{\infty} p_i u_{m,n-k_i} = c\Delta_1^2 u_{m-1,n+1} \tag{4.397}$$

for $m = 1,\dots,M$ and $n = 0,1,\dots$ with homogeneous Neumann boundary conditions (NBCs)

$$\Delta_1 u_{0,n} = \Delta_1 u_{M,n} = 0 \quad \text{for } n = 0,1,\dots \tag{4.398}$$

and initial conditions (ICs)

$$u_{m,i} = \mu_{m,i} \quad \text{for } m = 1,\dots,M,\ i = \dots,-1,0, \tag{4.399}$$

where Δ_1, Δ_1^2, and Δ_2 are defined as Section 3.7, $p_i, q_j \in R$, $k_i, r_j \in N_0$ for $i, j = 1,2,\dots$, $\mu_{m,i} \in R$ for $m = 1,\dots,M$ and $i = \dots,-1,0$, $c \geq 0$. Throughout this section, we assume that $P = \sum_{i=1}^{\infty} p_i > 0$, $P^* = \sum_{i=1}^{\infty} |p_i|$, $P' = \sum_{i=1}^{\infty} k_i|p_i|$, $P'' = \sum_{i=1}^{\infty} k_i^2|p_i|$, $Q^* = \sum_{j=1}^{\infty} |q_j|$, $Q' = \sum_{j=1}^{\infty} r_j|q_j|$, and $P, P^*, P', P'', Q^*, Q' < \infty$ and that

$$\|\mu\| = \sup\{|\mu_{m,i}| \quad \text{for } m = 1,\dots,M,\ i = \dots,-1,0\} < \infty. \tag{4.400}$$

For the sake of convenience, in proving the (unique) existence of solutions of (4.397) with the initial boundary conditions (4.398) and (4.399), we let $u_{m,i} = 0$ for $m < 0$, $m > M + 1$, and $i = 0,1,\dots$.

By a solution of (4.397)–(4.399), we mean a sequence $\{u_{m,n}\}$ which is defined for $m = 1,\dots,M$ and $n = 0,1,\dots$ and which satisfies (4.397), NBCs (4.398), and ICs (4.399).

By using the method similarly to Chapter 2 or simply by successive calculation, it is easy to show that (4.397) has a unique solution for given boundary and initial conditions which satisfies (4.400).

In the sequel, we only consider the solutions of (4.397) with the initial conditions satisfying (4.400).

We now give some definitions which will be needed in this section.

Definition 4.79. The zero solution of (4.397) is said to be attractive if every solution $\{u_{m,n}\}$ of (4.397) with ICs satisfying (4.400) has the property

$$\lim_{n\to\infty} u_{m,n} = 0 \quad \text{for } m = 1,\dots,M. \tag{4.401}$$

Definition 4.80. The zero solution of (4.397) is said to be L^2 stable if every solution $\{u_{m,n}\}$ of (4.397) with ICs satisfying (4.400) has the property

$$\sum_{n=0}^{\infty} u_{m,n}^2 < \infty \quad \text{for } m = 1,\dots,M. \tag{4.402}$$

It is easy to see that L^2 stability implies the attractivity.

Theorem 4.81. *Assume that*

$$Q^* + \frac{P}{2} + P' < 1. \tag{4.403}$$

Then the zero solution of (4.397) *is L^2 stable.*

Proof. It is easy to show that

$$\sum_{i=1}^{\infty} p_i u_{m,n-k_i} = P u_{m,n+1} - \Delta_2\left(\sum_{i=1}^{\infty} p_i \sum_{s=n-k_i}^{n} u_{m,s}\right). \tag{4.404}$$

Hence, we can rewrite (4.397) as follows:

$$\Delta_2\left(u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} - \sum_{i=1}^{\infty} p_i \sum_{s=n-k_i}^{n} u_{m,s}\right) = -P u_{m,n+1} + c\Delta_1^2 u_{m-1,n+1}. \tag{4.405}$$

Define a Liapunov sequence as follows:

$$V_n^{(1)} = \sum_{m=1}^{M}\left(u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} - \sum_{i=1}^{\infty} p_i \sum_{s=n-k_i}^{n} u_{m,s}\right)^2. \tag{4.406}$$

Then we have

$$\begin{aligned}\Delta V_n^{(1)} = \sum_{m=1}^{M} \big(- Pu_{m,n+1} + c\Delta_1^2 u_{m-1,n+1}\big) \\ \times \Bigg(u_{m,n+1} + u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n+1-r_j} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} - Pu_{m,n+1} \\ - 2\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} u_{m,s} - \sum_{i=1}^{\infty} p_i u_{m,n-k_i}\Bigg).\end{aligned} \tag{4.407}$$

Here, we define that $\sum_{i=m}^{n} * = 0$ if $m > n$.

For the estimation of the right-hand side of the above equality, let us consider

$$\begin{aligned}&- P\sum_{m=1}^{M} u_{m,n+1}\Bigg(u_{m,n+1} + u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n+1-r_j} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} \\ &\qquad - Pu_{m,n+1} - 2\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} u_{m,s} - \sum_{i=1}^{\infty} p_i u_{m,n-k_i}\Bigg) \\ &= -P\sum_{m=1}^{M} u_{m,n+1}\Bigg(u_{m,n+1} + u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n+1-r_j} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} \\ &\qquad - Pu_{m,n+1} - 2\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} u_{m,s} + u_{m,n+1} - u_{m,n} \\ &\qquad - \sum_{j=1}^{\infty} q_j u_{m,n+1-r_j} + \sum_{j=1}^{\infty} q_j u_{m,n-r_j} - c\Delta_1^2 u_{m-1,n+1}\Bigg) \\ &= \sum_{m=1}^{M} \Bigg(- 2Pu_{m,n+1}^2 + 2P\sum_{j=1}^{\infty} q_j u_{m,n+1} u_{m,n+1-r_j} + P^2 u_{m,n+1}^2 \\ &\qquad + 2P\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} u_{m,n+1} u_{m,s} + Pcu_{m,n+1}\Delta_1^2 u_{m-1,n+1}\Bigg) \\ &\le \sum_{m=1}^{M} \Bigg[- 2Pu_{m,n+1}^2 + P\sum_{j=1}^{\infty} |q_j| \big(u_{m,n+1}^2 + u_{m,n+1-r_j}^2\big) + P^2 u_{m,n+1}^2 \\ &\qquad + P\sum_{i=1}^{\infty} |p_i| \sum_{s=n+1-k_i}^{n} \big(u_{m,n+1}^2 + u_{m,s}^2\big) + Pcu_{m,n+1}\Delta_1^2 u_{m-1,n+1}\Bigg] \\ &= \sum_{m=1}^{M} \Bigg\{ - 2P\Big[1 - \frac{1}{2}\big(Q^* + P + P'\big)\Big] u_{m,n+1}^2 + P\sum_{j=1}^{\infty} |q_j| u_{m,n+1-r_j}^2 \\ &\qquad + P\sum_{i=1}^{\infty} |p_i| \sum_{s=n+1-k_i}^{n} u_{m,s}^2 + Pcu_{m,n+1}\Delta_1^2 u_{m-1,n+1}\Bigg\}.\end{aligned} \tag{4.408}$$

Similarly, we have the following inequality:

$$
\begin{aligned}
c \sum_{m=1}^{M} \Bigg(& u_{m,n+1} + u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n+1-r_j} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} - P u_{m,n+1} \\
& - 2\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} u_{m,s} - \sum_{i=1}^{\infty} p_i u_{m,n-k_i} \Bigg) \Delta_1^2 u_{m-1,n+1} \\
\le{}& 2c \sum_{m=1}^{M} u_{m,n+1} \Delta_1^2 u_{m-1,n+1} - 2c \sum_{j=1}^{\infty} q_j \sum_{m=1}^{M} u_{m,n+1-r_j} \Delta_1^2 u_{m-1,n+1} \\
& - Pc \sum_{m=1}^{M} u_{m,n+1} \Delta_1^2 u_{m-1,n+1} - 2c \sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} \sum_{m=1}^{M} u_{m,s} \Delta_1^2 u_{m-1,n+1}.
\end{aligned} \tag{4.409}
$$

Therefore, we obtain

$$
\begin{aligned}
\Delta V_n^{(1)} \le{}& -2P\left[1 - \frac{1}{2}(Q^* + P + P')\right] \sum_{m=1}^{M} u_{m,n+1}^2 + P \sum_{m=1}^{M} \sum_{j=1}^{\infty} |q_j| u_{m,n+1-r_j}^2 \\
& + P \sum_{m=1}^{M} \sum_{i=1}^{\infty} |p_i| \sum_{s=n+1-k_i}^{n} u_{m,s}^2 + 2c \sum_{m=1}^{M} u_{m,n+1} \Delta_1^2 u_{m-1,n+1} \\
& - c \sum_{j=1}^{\infty} q_j \sum_{m=1}^{M} u_{m,n+1-r_j} \Delta_1^2 u_{m-1,n+1} - 2c \sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} \sum_{m=1}^{M} u_{m,s} \Delta_1^2 u_{m-1,n+1}.
\end{aligned} \tag{4.410}
$$

By using a summation by parts formula and NBCs (4.398), (here we define $\Delta_1 u_{i,n} = 0$ for $i \le 0$ and $i \ge M+1$), we obtain

$$
2c \sum_{m=1}^{M} u_{m,n+1} \Delta_1^2 u_{m-1,n+1} = -2c \sum_{m=1}^{M} \left(\Delta_1 u_{m,n+1}\right)^2,
$$

$$
\begin{aligned}
- 2c \sum_{j=1}^{\infty} q_j & \sum_{m=1}^{M} u_{m,n+1-r_j} \Delta_1^2 u_{m-1,n+1} \\
&= 2c \sum_{j=1}^{\infty} q_j \sum_{m=1}^{M} \Delta_1 u_{m,n+1-r_j} \Delta_1 u_{m,n+1} \\
&\le c \sum_{m=1}^{M} \sum_{j=1}^{\infty} |q_j| \left[\left(\Delta_1 u_{m,n+1}\right)^2 + \left(\Delta_1 u_{m,n+1-r_j}\right)^2\right] \\
&= cQ^* \sum_{m=1}^{M} \left(\Delta_1 u_{m,n+1}\right)^2 + c \sum_{m=1}^{M} \sum_{j=1}^{\infty} |q_j| \left(\Delta_1 u_{m,n+1-r_j}\right)^2,
\end{aligned}
$$

$$\begin{aligned}
&- 2c\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} \sum_{m=1}^{M} u_{m,s}\Delta_1^2 u_{m-1,n+1} \\
&\quad = 2c\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} \sum_{m=1}^{M} \Delta_1 u_{m,s}\Delta_1 u_{m,n+1} \\
&\quad \le c\sum_{m=1}^{M}\sum_{i=1}^{\infty} |p_i| \sum_{s=n+1-k_i}^{n} \left[\left(\Delta_1 u_{m,s}\right)^2 + \left(\Delta_1 u_{m,n+1}\right)^2\right] \\
&\quad = cP'\sum_{m=1}^{M}\left(\Delta_1 u_{m,n+1}\right)^2 + c\sum_{m=1}^{M}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\left(\Delta_1 u_{m,s}\right)^2.
\end{aligned} \tag{4.411}$$

Using the above inequalities, we obtain

$$\begin{aligned}
\Delta V_n^{(1)} \le{}& -2P\left[1 - \frac{1}{2}\left(Q^* + P + P'\right)\right]\sum_{m=1}^{M} u_{m,n+1}^2 \\
&- 2c\left[1 - \frac{1}{2}\left(Q^* + P'\right)\right]\sum_{m=1}^{M}\left(\Delta_1 u_{m,n+1}\right)^2 \\
&+ P\sum_{m=1}^{M}\sum_{j=1}^{\infty}|q_j|\,u_{m,n+1-r_j}^2 + P\sum_{m=1}^{M}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n} u_{m,s}^2 \\
&+ c\sum_{m=1}^{M}\sum_{j=1}^{\infty}|q_j|\left(\Delta_1 u_{m,n+1-r_j}\right)^2 + c\sum_{m=1}^{M}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\left(\Delta_1 u_{m,s}\right)^2.
\end{aligned} \tag{4.412}$$

Now, define another Liapunov sequence as follows:

$$\begin{aligned}
V_n^{(2)} = \sum_{m=1}^{M}\Bigg[&P\sum_{j=1}^{\infty}|q_j|\sum_{s=n+1-r_j}^{n} u_{m,s}^2 + P\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\sum_{t=s}^{n} u_{m,t}^2 \\
&+ c\sum_{j=1}^{\infty}|q_j|\sum_{s=n+1-r_j}^{n}\left(\Delta_1 u_{m,s}\right)^2 + c\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\sum_{t=s}^{n}\left(\Delta_1 u_{m,t}\right)^2\Bigg].
\end{aligned} \tag{4.413}$$

Then, we obtain

$$\begin{aligned}\Delta V_n^{(2)} = \sum_{m=1}^{M}\Bigg\{&P\sum_{j=1}^{\infty}|q_j|\left(u_{m,n+1}^2-u_{m,n+1-r_j}^2\right)+P\sum_{i=1}^{\infty}|p_i|\left(k_iu_{m,n+1}^2-\sum_{s=n+1-k_i}^{n}u_{m,s}^2\right)\\&+c\sum_{j=1}^{\infty}|q_j|\left[\left(\Delta_1u_{m,n+1}\right)^2-\left(\Delta_1u_{m,n+1-r_j}\right)^2\right]\\&+c\sum_{i=1}^{\infty}|p_i|\left[k_i\left(\Delta_1u_{m,n+1}\right)^2-\sum_{s=n+1-k_i}^{n}\left(\Delta_1u_{m,s}\right)^2\right]\Bigg\}\\=\sum_{m=1}^{M}\Bigg[&PQ^*u_{m,n+1}^2+PP'u_{m,n+1}^2+cQ^*\left(\Delta_1u_{m,n+1}\right)^2+cP'\left(\Delta_1u_{m,n+1}\right)^2\\&-P\sum_{j=1}^{\infty}|q_j|u_{m,n+1-r_j}^2-P\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}u_{m,s}^2\\&-c\sum_{j=1}^{\infty}|q_j|\left(\Delta_1u_{m,n+1-r_j}\right)^2-c\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\left(\Delta_1u_{m,s}\right)^2\Bigg].\end{aligned}\tag{4.414}$$

Finally, we take the Liapunov sequence as $V_n = V_n^{(1)} + V_n^{(2)}$. By using (4.403), we finally obtain

$$\Delta V_n \leq -2P\left(1-Q^*-\frac{P}{2}-P'\right)\sum_{m=1}^{M}u_{m,n+1}^2. \tag{4.415}$$

Therefore, $\{V_n\}$ is decreasing and there exists a nonnegative limit $V_0 = \lim_{n\to\infty} V_n$. Now, summing both sides of (4.415) from $n = 0$ to $n = \infty$, we have

$$2P\left(1-Q^*-\frac{P}{2}-P'\right)\sum_{n=0}^{\infty}\sum_{m=1}^{M}u_{m,n+1}^2 \leq V_0. \tag{4.416}$$

Hence,

$$\sum_{n=0}^{\infty}\sum_{m=1}^{M}u_{m,n}^2 \leq \sum_{m=1}^{M}\mu_{m,0}^2+\frac{V_0}{2P\left(1-Q^*-P/2-P'\right)}<\infty. \tag{4.417}$$

The proof is complete. □

Remark 4.82. Let $u_{m,n}$ be independent of m, writing $x_n = u_{m,n}$, $q_j = k_i = 0$ for $i, j = 1, 2, \dots$ and $c = 0$. Then (4.397) becomes an ordinary difference equation

$$\Delta x_n + Px_n = 0 \quad \text{for } n = 0, 1, \dots \tag{4.418}$$

and (4.403) becomes

$$\frac{P}{2} < 1. \tag{4.419}$$

One can easily prove that the condition (4.419) is a necessary and sufficient condition for L^2 stability of (4.418). In fact, its solutions $\{x_n\}$ with IC $x_i = \mu_i$ for $i = \dots, -1, 0$ satisfying $\|\mu\| = \sup\{|\mu_i| \text{ for } i = \dots, -1, 0\} < \infty$ has the property $\sum_{n=0}^{\infty} |x_n| < \infty$. Therefore, in this sense, the condition (4.403) is a "sharp" condition.

As a special case, we consider a linear parabolic Volterra difference equation

$$\Delta_2 u_{m,n} + \sum_{i=1}^{\infty} p_i u_{m,n-k_i} = c\Delta_1^2 u_{m-1,n+1} \tag{4.420}$$

for $m = 1, \dots, M$ and $n = 0, 1, \dots,$ with NBCs (4.398) and ICs (4.399). By Theorem 4.81, we have the following conclusion.

Corollary 4.83. *If*

$$\frac{P}{2} + P' < 1, \tag{4.421}$$

then the zero solution of (4.420) *is* L^2 *stable.*

The above argument can be used to the nonlinear parabolic Volterra difference equations.

4.6. Systems of nonlinear Volterra difference equations with diffusion and infinite delay

We consider the r-dimensional Euclidean space R^r. For $x = (x_1, x_2, \dots, x_r)^T \in R^r$, we define its norm $\|x\| = \max_{i\in I} |x_i|$, where $I = \{1, 2, \dots, r\}$. In R^r, we introduce a cone $P = \{x \mid x_i \geq 0, i \in I\}$. Then it is a solid cone in R^r. It is easy to show that P is normal, regular, minimal, strong minimal, and regenerated (see Amann [9]). For two elements x and $y = (y_1, y_2, \dots, y_r)^T$ in P, we introduce a partial ordering $\leq$ such that $x <$ (or $=$)y if and only if $x_i <$ (or $=$)y_i for $i \in I$ and $x \leq y$ means that $x_i \leq y_i$ for $i \in I$. So, $(R^r, \leq)$ becomes a partial-ordered Banach space. In R^r, we also define an operation of multiplication $\otimes$ by $x \otimes y = (x_1y_1, x_2y_2, \dots, x_ry_r)^T$. In this way, $(R^r, +, \otimes)$ will be a partial ordered commutative ring by installing both this operation $\otimes$ and the ordinary addition $+$ with the zero element $0 = (0, 0, \dots, 0)^T$ and the unit element $u = (1, 1, \dots, 1)^T$. Define an ordered interval $[\cdot, \cdot]$ in R^r by $[x, y] = \{z \in R^r \mid x \leq z \leq y\}$.

In the $r \times r$-dimensional matrix space $R^{r\times r}$, we also introduce a partial ordering $\leq$. If $X = (x_{i,j})_{r\times r}$ and $Y = (y_{i,j})_{r\times r}$ are two elements in $R^{r\times r}$, then define that $X < (\text{or } =)Y$ if and only if $x_{i,j} < (\text{or } =)y_{i,j}$ for $i, j \in I$ and $X \leq Y$ means that $x_{i,j} \leq y_{i,j}$ for $i, j \in I$. Therefore, $R^{r\times r}$ also becomes a partial-ordered Banach space.

Consider the systems of nonlinear Volterra difference equations of the population model with diffusion and infinite delays

$$\Delta_2 u_{m,n} = A\Delta_1^2 u_{m-1,n+1} + u_{m,n} \otimes \left(b - Cu_{m,n} - \sum_{i=0}^{\infty} D_i u_{m,n-i}\right) \tag{4.422}$$

for $(m,n) \in \Omega \times N_0 \triangleq \{1,2,\dots,M_1\} \times \cdots \times \{1,2,\dots,M_s\} \times \{0,1,\dots\}$, where Δ_1 and Δ_2 are forward partial difference operators, Δ_1^2 is discrete Laplacian operator, $A, C > (0)_{r\times r}$ are diagonal matrices, $b \in R^r$, and $b > 0$, $u_{\cdot,\cdot} \in R^r$ is a double vector sequence (only in form), $D_0 = (0)_{r\times r}$ and $D_i \in R^{r\times r}$ for $i \in N_0$.

Together with (4.422), we consider homogeneous Neumann boundary condition

$$\Delta_N u_{m-1,n+1} = 0 \quad \text{for } (m,n) \in \partial\Omega \times N_0 \tag{4.423}$$

and initial condition

$$u_{m,j} = \phi_{m,j} \quad \text{for } (m,n) \in \Omega \times N^{(0)} \triangleq \Omega \times \{\dots,-1,0\}, \tag{4.424}$$

where Δ_N is the normal difference, $\partial\Omega$ is the boundary of Ω, and $\phi_{m,j} \in P$ for $(m,n) \in \Omega \times N^{(0)}$.

By a solution of (4.422)–(4.424), we mean a double vector sequence $\{u_{m,n}\}$, which is defined on $(m,n) \in \Omega \times N \triangleq \Omega \times N_0 \cup N^{(0)}$, satisfies (4.422), (4.423), and (4.424), respectively, when $(m,n) \in \Omega \times N_0$, $(m,n) \in \partial\Omega \times N_0$, and $(m,j) \in \Omega \times N^{(0)}$.

For any given initial and boundary condition (4.423) and (4.424), we can show that the initial and boundary value problem (4.422)–(4.424) has a unique solution.

We suppose that

$$\begin{gathered} \sum_{i=0}^{\infty} |D_i| = D < \infty, \\ 0 < \|\phi\| = \sup_{(m,j)\in\Omega\times N^{(0)}} \phi_{m,j} < \infty. \end{gathered} \tag{4.425}$$

We write throughout this section that

$$D_n = \sum_{i=0}^{n} |D_i|, \quad \delta_n = \sum_{i=0}^{n} D_i, \quad D_n^{\pm} = \frac{D_n \pm \delta_n}{2} \quad \text{for } n \in N_0. \tag{4.426}$$

Then D_n, $D_n^{\pm}$ are all nonnegative, nondecreasing, and bounded above by D.

Since P is regular, we can let $D^{\pm} = \lim_{n\to\infty} D_n^{\pm}$. It is easy to see that

$$\begin{aligned} D_n^+ + D_n^- &= D_n, \qquad D_n^+ - D_n^- = \delta_n, \\ D^+ + D^- &= D, \qquad D^+ - D^- = \delta = \sum_{i=0}^{\infty} D_i. \end{aligned} \tag{4.427}$$

Assume that

$$Cu > D^- u. \tag{4.428}$$

From Berman and Plemmons [20], we know that $C - D^-$ is a nonsingular and inverse-positive Metzlerian matrix, that is, $C-D^-$ is invertible and $\det(C-D^-)^{-1} > 0$. Then $(C - D^-)^{-1}b > 0$. Since $C + \delta > C - D^-$, we know from Metzlerian matrix theory that $C + \delta$ is invertible and $\det(C + \delta)^{-1} > 0$.

In addition, we let

$$b - D^+ (C - D^-)^{-1} b > 0, \tag{4.429}$$

$$p = \max \{ (C - D^-)^{-1} b, \|\phi\| \}. \tag{4.430}$$

It is obvious that the nonlinear Volterra difference equation of the population model

$$\Delta x_n = x_n \left(b - c x_n - \sum_{i=0}^{\infty} d_i x_{n-i} \right) \quad \text{for } n \in N_0 \tag{4.431}$$

is a special case when $r = 1$ and without diffusion, where Δ is the forward difference operator.

It is easy to show that (4.422) has only two equilibrium points $u_{m,n} \equiv 0$ and $u_{m,n} \equiv (C + \delta)^{-1} b$. The purpose of this section is to give a sufficient condition for the attractivity of the positive equilibrium solution $u_{m,n} \equiv (C+\delta)^{-1} b$ of (4.422) by using the method of lower and upper solutions and monotone iterative techniques.

Lemma 4.84. *Let* (4.425), (4.428), (4.429), *and* (4.430) *hold. Suppose that* $\{u_{m,n}\}$ *is the unique solution of* (4.422)–(4.424). *Then*

$$u_{m,n} \in [0, p] \quad \textit{for } (m, n) \in \Omega \times N_0. \tag{4.432}$$

Proof. Consider the initial boundary value problems

$$\begin{aligned}
\Delta_2 v_{m,n} &\le A\Delta_1^2 v_{m-1,n+1} + v_{m,n} \otimes \left(b - Cv_{m,n} - \sum_{i=0}^{\infty} D_i v_{m,n-i}\right) \quad \text{for } (m,n) \in \Omega \times N_0,\\
\Delta_N v_{m-1,n+1} &= 0 \le 0 \quad \text{for } (m,n) \in \partial\Omega \times N_0,\\
v_{m,j} &= 0 \le \phi_{m,j} \quad \text{for } (m,n) \in \Omega \times N^{(0)};
\end{aligned} \tag{4.433}$$

$$\begin{aligned}
\Delta_2 w_{m,n} &\ge A\Delta_1^2 w_{m-1,n+1} + w_{m,n} \otimes \left(b - Cw_{m,n} - \sum_{i=0}^{\infty} D_i w_{m,n-i}\right) \quad \text{for } (m,n) \in \Omega \times N_0,\\
\Delta_N w_{m-1,n+1} &= 0 \ge 0 \quad \text{for } (m,n) \in \partial\Omega \times N_0,\\
w_{m,j} &= p \ge \phi_{m,j} \quad \text{for } (m,n) \in \Omega \times N^{(0)}.
\end{aligned} \tag{4.434}$$

Since

$$b - (C+\delta)p \le b - Cp + D^- p = b - (C - D^-)p \le 0, \tag{4.435}$$

it is easy to see that $v \equiv 0$ and $w_{m,n} \equiv p$ are, respectively, solutions of (4.433) and (4.434), that is, a pair of lower and upper solutions of (4.422)–(4.424). Therefore, (4.432) holds. This completes the proof. □

Lemma 4.85. *Assume that* (4.425), (4.428)–(4.430) *hold. Suppose that* $\{p_n^{(1)}\}$ *is the unique solution of the Cauchy problem*

$$\begin{aligned}
\Delta p_n^{(1)} &= p_n^{(1)} \otimes \left(b - Cp_n^{(1)} + D^- p\right) \quad \textit{for } n \in N_0,\\
p_j^{(1)} &= p \quad \textit{for } j \in N^{(0)}.
\end{aligned} \tag{4.436}$$

Then $\{p_n^{(1)}\}$ *is nonincreasing and*

$$p_n^{(1)} \in \left[C^{-1}(b + D^- p), p\right] \quad \textit{for } n \in N_0. \tag{4.437}$$

Proof. Consider the Cauchy problems

$$\begin{aligned}
\Delta v_n^{(1)} &\le v_n^{(1)} \otimes \left(b - Cv_n^{(1)} + D^- p\right) \quad \text{for } n \in N_0,\\
v_j^{(1)} &= C^{-1}(b + D^- p) \le p \quad \text{for } j \in N^{(0)};
\end{aligned} \tag{4.438}$$

$$\begin{aligned}
\Delta w_n^{(1)} &\ge w_n^{(1)} \otimes \left(b - Cw_n^{(1)} + D^- p\right) \quad \text{for } n \in N_0,\\
w_j^{(1)} &= p \ge p \quad \text{for } j \in N^{(0)}.
\end{aligned} \tag{4.439}$$

It is easy for one to know that $v_n^{(1)} \equiv C^{-1}(b + D^- p)$ and $w_n^{(1)} \equiv p$ are, respectively, solutions of (4.438) and (4.439), that is, a pair of lower and upper solutions of (4.436). So, (4.437) holds.

By (4.437), we have that $\Delta p_n^{(1)} \le 0$. Hence, $\{p_n^{(1)}\}$ is nonincreasing. The proof is thus complete. □

Lemma 4.86. *Let* (4.425), (4.428)–(4.430) *hold. Suppose that* $\{u_{m,n}\}$ *and* $\{p_n^{(1)}\}$ *are, respectively, the unique solutions of* (4.422), (4.423), (4.424) *and* (4.436). *Then*

$$u_{m,n} \in [0, p_n^{(1)}] \quad \text{for } (m,n) \in \Omega \times N_0. \tag{4.440}$$

Proof. Let $J^\pm$ satisfy that $J^+ \cup J^- = N_0$ and let $J^+ \cap J^- = \varnothing$, the empty set, and be such that

$$D_i \ge (0)_{r\times r} \quad \text{for } i \in J^+, \qquad D_i < (0)_{r\times r} \quad \text{for } i \in J^-. \tag{4.441}$$

Write $\delta^\pm = \sum_{i\in J^\pm} D_i$. Then we must have

$$\delta^+ = D^+, \qquad -\delta^- = D^-. \tag{4.442}$$

Hence, from (4.437), we have

$$\begin{gathered} -\sum_{i=0}^{\infty} D_i p_{n-i}^{(1)} = -\sum_{i\in J^+} D_i p_{n-i}^{(1)} - \sum_{i\in J^-} D_i p_{n-i}^{(1)} \le -\sum_{i\in J^-} D_i p_{n-i}^{(1)} \le -\delta^- p = D^- p, \\ b - Cp_n^{(1)} - \sum_{i=0}^{\infty} D_i p_{n-i}^{(1)} \le b - Cp_n^{(1)} + D^- p \quad \text{for } n \in N_0. \end{gathered} \tag{4.443}$$

Therefore, $w_{m,n} \equiv p_n^{(1)}$ is a solution of (4.434), and (4.440) holds. Thus the proof is complete. □

For the regularity of P, we can let $p^{(1)} = \lim_{n\to\infty} p_n^{(1)}$. By virtue of (4.436), we can obtain $p^{(1)} = C^{-1}(b + D^- p)$. It follows that

$$\limsup_{n\to\infty} \max_{m\in\Omega} u_{m,n} \le p^{(1)}. \tag{4.444}$$

So, for any $\epsilon = (\epsilon, \dots, \epsilon)^T > 0$, there exist $n_1 > 0$ and $n_2 > n_1$ such that

$$u_{m,n} < p^{(1)} + \epsilon \quad \text{for } n \in N_{n_1}, \tag{4.445}$$

$$(0)_{r\times r} \le D^- - D^-_{n-n_1-1} < (\epsilon)_{r\times r} \quad \text{for } n \in N_{n_2}. \tag{4.446}$$

Lemma 4.87. *Let* (4.425), (4.428)–(4.430) *hold. Suppose that* $\{p_n^{(2)}\}$ *is the unique solution of the Cauchy problem*

$$\Delta p_n^{(2)} = p_n^{(2)} \otimes \left(b - Cp_n^{(2)} + D^-(p_1+\epsilon) + \epsilon \otimes p\right) \quad \textit{for } n \in N_{n_2},$$
$$p_j^{(2)} = p_1 + \epsilon \quad \textit{for } j \in N^{(n_2)}, \tag{4.447}$$

where $N^{(n_2)} = \{\dots, n_2-1, n_2\}$. *Then* $p_n^{(2)}$ *is nonincreasing and*

$$u_{m,n} \in \left[0, p_n^{(2)}\right] \quad \textit{for } (m,n) \in \Omega \times N_{n_2}. \tag{4.448}$$

Proof. If (4.448) is not true, then there exist $m_3 \in \Omega$ and $n_3 > n_2$ such that $u_{m,n} \le p_n^{(2)}$ for $n_2 \le n < n_3$ and $m \in \Omega$ and $u_{m_3,n_3} > p_n^{(2)}$.

Let $x_{m,n} = u_{m,n} - p_n^{(2)}$. Then $x_{m,n} \le 0$ for $n_2 \le n < n_3$ and $m \in \Omega$ and

$$x_{m_3,n_3} > 0. \tag{4.449}$$

We can derive, from (4.447),

$$A\Delta_1^2 x_{m-1,n+1} - \Delta_2 x_{m,n} + y_{m,n} \otimes x_{m,n} = z_{m,n} \quad \text{for } (m,n) \in \Omega \times N_{n_2}, \tag{4.450}$$

where

$$y_{m,n} = b - Cu_{m,n} - \sum_{i=0}^{\infty} D_i u_{m,n-i} - Cp_n^{(2)} \quad \text{for } (m,n) \in \Omega \times N_{n_2},$$
$$z_{m,n} = p_n^{(2)} \otimes \left(D^-(p_1+\epsilon) + \sum_{i=0}^{\infty} D_i u_{m,n-i} + \epsilon \otimes p\right) \quad \text{for } (m,n) \in \Omega \times N_{n_2}. \tag{4.451}$$

It is easy to show that $y_{m,n}$ is bounded. We will see in the following that $z_{m,n} \ge 0$. Indeed, from (4.445) and (4.446), we have

$$\begin{aligned} -\sum_{i=0}^{\infty} D_i u_{m,n-i} &= -\sum_{i=0}^{\infty} (\Delta\delta_{i-1}) u_{m,n-i} = -\sum_{i=0}^{\infty} (\Delta D_{i-1}^+) u_{m,n-i} + \sum_{i=0}^{\infty} (\Delta D_{i-1}^-) u_{m,n-i} \\ &\le \sum_{i=0}^{n-n_1-1} (\Delta D_{i-1}^-) u_{m,n-i} + \sum_{i=n-n_1}^{\infty} (\Delta D_{i-1}^-) u_{m,n-i} \\ &\le D_{n-n_1-1}^-(p_1+\epsilon) + \left(D^- - D_{n-n_1-1}^-\right) p \le D^-(p_1+\epsilon) + (\epsilon)_{r\times r} p. \end{aligned} \tag{4.452}$$

So, by (4.451) and (4.452), we have

$$z_{m,n} \ge 0 \quad \text{for } (m,n) \in \Omega \times N_{n_2}. \tag{4.453}$$

It follows, from (4.450),

$$\Delta_2 x_{m,n} \leq A\Delta_1^2 x_{m-1,n+1} + y_{m,n} \otimes x_{m,n}. \tag{4.454}$$

Consider the initial boundary problems

$$\begin{aligned} \Delta_2 v_{m,n} &= A\Delta_1^2 v_{m-1,n+1} + y_{m,n} \otimes v_{m,n} \quad \text{for } (m,n) \in \Omega \times N_{n_2}, \\ \Delta_N v_{m-1,n+1} &= 0 \quad \text{for } (m,n) \in \partial\Omega \times N_{n_2}, \\ v_{m,n_2} &= 0 \quad \text{for } m \in \Omega, \end{aligned} \tag{4.455}$$

$$\begin{aligned} \Delta_2 x_{m,n} &\leq A\Delta_1^2 x_{m-1,n+1} + y_{m,n} \otimes x_{m,n} \quad \text{for } (m,n) \in \Omega \times N_{n_2}, \\ \Delta_N x_{m-1,n+1} &\leq 0 \quad \text{for } (m,n) \in \partial\Omega \times N_{n_2}, \\ x_{m,n_2} &\leq 0 \quad \text{for } m \in \Omega. \end{aligned} \tag{4.456}$$

Obviously, $v_{m,n} \equiv 0$ is the unique solution of (4.455). Comparing (4.455) with (4.456), we know that $x_{m,n} \leq 0$ for $(m,n) \in \Omega \times N_{n_2}$. But, this contradicts (4.449). Therefore, (4.448) holds.

Similar to the proof of Lemma 4.85, we can easily know that $p_n^{(2)}$ is nonincreasing, which completes the proof. □

Remark 4.88. In fact, we can directly use the maximum principle (see Cheng [29]) to obtain the contradiction.

We can obtain from (4.447) and the regularity of P that

$$\lim_{n\to\infty} p_n^{(2)} = C^{-1}\left(b + D^-(p_1 + \epsilon) + \epsilon \otimes p\right). \tag{4.457}$$

Therefore,

$$\limsup_{n\to\infty} \max_{m\in\Omega} u_{m,n} \leq C^{-1}\left(b + D^-(p_1 + \epsilon) + \epsilon \otimes p\right). \tag{4.458}$$

Because ϵ is arbitrary, we have

$$\limsup_{n\to\infty} \max_{m\in\Omega} u_{m,n} \leq C^{-1}\left(b + D^- p_1\right) \triangleq p_2. \tag{4.459}$$

Define a sequence $\{p_\ell\}$ as follows:

$$p_\ell = C^{-1}\left(b + D^- p_{\ell-1}\right) \quad \text{for } \ell \in Z^+(1), \qquad p_0 = p. \tag{4.460}$$

Lemma 4.89. *Let* (4.425), (4.428)–(4.430) *hold. Suppose that* $\{p_\ell\}$ *is defined by* (4.460). *Then* $\{p_\ell\}$ *is nonincreasing and*

$$\left(C - D^-\right)^{-1} b \in [0, p_\ell] \quad \text{for } \ell \in N_0. \tag{4.461}$$

Proof. We rewrite (4.460) as follows:

$$\Delta p_\ell = C^{-1}D^{-}\Delta p_{\ell-1} \quad \text{for } \ell \in N_1. \tag{4.462}$$

We know from $\Delta p_0 = p_1 - p \le 0$ that $\Delta p_\ell \le 0$ for all $\ell \in N_0$. That is, $\{p_\ell\}$ is nonincreasing.

Noting that $p \ge (C - D^-)^{-1}b$, we have from the equations in (4.460) and (4.437) that $Cp_1 = b + D^- p \ge b + D^- p_1$. Hence, $p_1 \ge (C - D^-)^{-1}b$. By induction, we obtain (4.461). This completes the proof. □

Because P is regular, we let $\gamma = \lim_{\ell\to\infty} p_\ell$. From (4.460), we have $\gamma = C^{-1}(b + D^-\gamma)$. We can solve $\gamma = (C - D^-)^{-1}b$.

Repeating the above procedure, we can show that

$$\limsup_{n\to\infty} \max_{m\in\Omega} u_{m,n} \le \gamma. \tag{4.463}$$

From (4.429), we have that $b > D^+\gamma$. So, we can select an $\epsilon_0 > 0$ such that

$$b > D^+(\gamma + \epsilon_0) + \epsilon_0 \otimes p. \tag{4.464}$$

Let $0 < \epsilon < \epsilon_0$. By (4.452), there exist $n_4 > n_3$ and $n_5 > n_4$ such that

$$\begin{gathered} u_{m,n} < \gamma + \epsilon \quad \text{for } (m,n) \in \Omega \times N_{n_4}, \\ (0)_{r\times r} \le D^+ - D^+_{n-n_4-1} < (\epsilon)_{r\times r} \quad \text{for } n \in N_{n_5}. \end{gathered} \tag{4.465}$$

From (4.423), Lemma 4.84 and the maximum principle (see Cheng [29]), we know that $u_{m,n} > 0$ for $(m,n) \in \Omega \times N_0$ and can select an $\eta > 0$ such that $\min_{m\in\Omega} u_{m,n_5} \ge 2\eta$.

Consider the Cauchy problem

$$\begin{gathered} \Delta q_n = q_n \otimes \big(b - Cq_n - D^+(\gamma+\epsilon) - \epsilon \otimes p\big) \quad \text{for } n \in N_{n_5}, \\ q_j = \eta \quad \text{for } j \in N_{n_5}. \end{gathered} \tag{4.466}$$

Repeating a similar argument of the above, we can obtain that

$$\begin{gathered} q_n < u_{m,n} \quad \text{for } (m,n) \in \Omega \times N_{n_5}, \\ \lim_{n\to\infty} q_n = C^{-1}\big(b - D^+(\gamma+\epsilon) - \epsilon\big) \otimes p. \end{gathered} \tag{4.467}$$

Consequently, we have

$$\liminf_{n\to\infty} \min_{m\in\Omega} u_{m,n} \ge C^{-1}\big(b - D^+\gamma\big) \tag{4.468}$$

for $\epsilon > 0$ being arbitrary.

Define a pair of coupled sequences $\{\mu_k\}$ and $\{\nu_k\}$ as follows:

$$\begin{aligned} C\mu_k &= b - D^+\nu_{k-1} + D^-\mu_{k-1} \quad \text{for } k \in N_1, \\ C\nu_k &= b + D^-\nu_{k-1} - D^+\mu_{k-1} \quad \text{for } k \in N_1, \\ \nu_0 &= (C - D^-)^{-1}b, \qquad \mu_0 = C^{-1}(b - D^+\nu_0). \end{aligned} \tag{4.469}$$

Lemma 4.90. *Let* (4.425), (4.428)–(4.430) *hold. Suppose that the pair* $\{\mu_k\}$ *and* $\{\nu_k\}$ *is defined by* (4.469). *Then*

$$[\mu_0, \nu_0] \supseteq [\mu_1, \nu_1] \supseteq \cdots \supseteq [\mu_k, \nu_k] \supseteq \cdots \quad \textit{for } k \in N_0, \tag{4.470}$$

$$\lim_{k\to\infty} \mu_k = \lim_{k\to\infty} \nu_k = (C + \delta)^{-1}b. \tag{4.471}$$

Proof. Because

$$\begin{gathered} C\mu_1 \geq b - D^+\nu_0 = C\mu_0, \qquad C\nu_0 = (C - D^-)\nu_0 + D^-\nu_0 = b + D^-\nu_0 \geq C\nu_1, \\ C\nu_0 \geq (C - D^-)\nu_0 = b \geq b - D^+\nu_0 = C\mu_0, \end{gathered} \tag{4.472}$$

so we have

$$[\mu_0, \nu_0] \supseteq [\mu_1, \nu_1]. \tag{4.473}$$

We can get (4.470) by induction.

By virtue of the regularity of P, we can let $\mu = \lim_{k\to\infty} \mu_k$ and $\nu = \lim_{k\to\infty} \nu_k$. Then we get

$$\begin{aligned} C\mu &= b - D^+\nu + D^-\mu, \\ C\nu &= b + D^-\nu - D^+\mu. \end{aligned} \tag{4.474}$$

Subtracting the two equalities in (4.474), we obtain

$$C(\mu - \nu) = (D^+ + D^-)(\mu - \nu) = D(\mu - \nu). \tag{4.475}$$

So, $(C - D)(\mu - \nu) = 0$.

Since

$$(C - D)\nu_0 = (C - D^+ - D^-)(C - D^-)^{-1}b = b - D^+(C - D^-)^{-1}b > 0 \tag{4.476}$$

from (4.429), we have from the properties of Metzlerian matrices that $\det(C - D)^{-1} > 0$. Therefore, $\mu = \nu$.

It follows from (4.474) that

$$C\mu = b - D^+\mu + D^-\mu = b - \delta\mu \quad \text{or} \quad (C+\delta)\mu = b. \tag{4.477}$$

This leads to (4.471). The proof is thus complete. □

Lemma 4.91. *Let* (4.425), (4.428)–(4.430) *hold. Suppose that the pair* $\{\mu_k\}$ *and* $\{\nu_k\}$ *is defined by* (4.469). *Then*

$$\Big[\liminf_{n\to\infty}\min_{m\in\Omega} u_{m,n}, \limsup_{n\to\infty}\max_{m\in\Omega} u_{m,n}\Big] \subseteq [\mu_k, \nu_k] \quad \textit{for } k \in N_0. \tag{4.478}$$

Proof. From the above, (4.478) holds for $k = 0$.

Take an $\epsilon_1 > 0$ such that $\epsilon_1 < \mu_0$ and

$$b > D^+(\nu_0+\epsilon_1) - D^-(\mu_0-\epsilon_1) + 2\epsilon_1 \otimes p. \tag{4.479}$$

For $0 < \epsilon < \epsilon_1$, there exist $n_6 > n_5$ and $n_7 > n_6$ such that

$$\begin{aligned} &\mu_0 - \epsilon < u_{m,n} < \nu_0 + \epsilon \quad \text{for } (m,n) \in \Omega \times N_{n_6}, \\ &(0)_{r\times r} \le D - D_{n-n_6-1} < (\epsilon)_{r\times r} \quad \text{for } n \in N_{n_7}. \end{aligned} \tag{4.480}$$

Now, we consider the Cauchy problems

$$\begin{aligned} \Delta\overline{p}_n &= \overline{p}_n \otimes \big(b - C\overline{p}_n + D^-(\nu_0+\epsilon) - D^+(\mu_0-\epsilon) + 2\epsilon \otimes p\big), \quad n \in N_{n_7}, \\ \overline{p}_j &= \nu_0 + \epsilon, \quad j \in N^{(n_7)}; \end{aligned} \tag{4.481}$$

$$\begin{aligned} \Delta\overline{q}_n &= \overline{q}_n \otimes \big(b - C\overline{q}_n - D^+(\nu_0+\epsilon) + D^-(\mu_0-\epsilon) - 2\epsilon \otimes p\big), \quad n \in N_{n_7}, \\ \overline{q}_j &= \mu_0 - \epsilon, \quad j \in N^{(n_7)}. \end{aligned} \tag{4.482}$$

Similar to the above argument, we can obtain

$$\begin{aligned} &\overline{q}_n < u_{m,n} < \overline{p}_n \quad \text{for } (m,n) \in \Omega \times N_{n_7}, \\ &\lim_{n\to\infty}\overline{p}_n = C^{-1}\big(b + D^-(\nu_0+\epsilon) - D^+(\mu_0-\epsilon) + 2\epsilon \otimes p\big), \\ &\lim_{n\to\infty}\overline{q}_n = C^{-1}\big(b - D^+(\nu_0+\epsilon) + D^-(\mu_0-\epsilon) - 2\epsilon \otimes p\big). \end{aligned} \tag{4.483}$$

Letting $\epsilon \to 0$, we know that (4.478) holds for $k = 1$.

Again, by repeating the above process, we have that (4.478) holds. □

Using the above seven lemmas, together with the property that P is normal, we have the following main result.

Theorem 4.92. *Let* (4.425), (4.428)–(4.430) *hold. Assume that* $\{u_{m,n}\}$ *is the unique solution of* (4.422), (4.423), *and* (4.424). *Then*

$$\lim_{n\to\infty, m\in\Omega} u_{m,n} = (C+\delta)^{-1}b. \tag{4.484}$$

Remark 4.93. It is well known that (4.422) describes the growth of r-species alive in Ω that the densities of the r-populations at place m and time n are $u_{m,n}$ and the summation represents the effects of the past history on the present growth rate in mathematical ecology. Therefore, we can only consider the case $\|\phi\| > 0$. If it is not the case, this will mean these species do not exist. The condition $\|\phi\| < \infty$ means that the densities of these species should be finite in practice. Equation (4.484) means that the growth of these species will go to an equilibrium state under ordinary conditions.

4.7. Notes

Theorems 4.1 and 4.2 are taken from Tian and Zhang [140]. Theorems 4.8,4.10, 4.11, 4.18, and 4.20 are taken from Tian et al. [138]. Theorems 4.12 and 4.13 are adopted from Tian and Zhang [142]. Theorem 4.16 is taken from Zhang and Deng [166]. The material of Section 4.2.2 is taken from Tian and Zhang [139]. The material of Section 4.3 is adopted from Zhang and Deng [165]. Lemma 4.39 is based on Banaś and Goebal [17]. The material of Section 4.4 is taken from Tian and Chen [134, 135]. The material of Section 4.5 is taken from Shi et al. [127]. The material of Section 4.6 is adopted from Shi [124]. The global attractivity of a class IBVP of delay partial difference equations can be seen from Zhang and Yu [191].

5 Spatial chaos

5.1. Introduction

In 1975, Li and Yorke introduced the first precise mathematical definition of chaos and obtained the well-known result, that is, "period 3 implies chaos." The theory of chaos of dynamic systems has grown at an accelerated pace in the past thirty years. There are several different definitions of chaos in the literature. In this chapter, we will describe some of the recent developments in chaos of partial difference equations.

The iteration problem of spatially multivariable sequence is not only a heart problem of spatial orbits of the motion in research progress but also an important concept. In Section 5.2, an iterative method of the spatial sequence is given. Then, spatially k-periodic orbit is produced and a basic criterion of spatially chaotic behavior in the sense of Li-York is obtained.

In Section 5.3, we establish the relation between chaos of certain partial difference equations and chaos of discrete dynamical system in complete metric spaces in the sense of Devaney.

In Section 5.4, we discuss discrete dynamical systems governed by continuous maps in complete metric spaces and present some criteria of chaos.

5.2. On spatial periodic orbits and spatial chaos

In this section, we introduce a constructive technique for generating spatial periodic orbits and then give a criterion of spatial chaos for the following 2D nonlinear system:

$$x_{m+1,n} + ax_{m,n+1} = f\big[(1+a)x_{mn}\big], \tag{5.1}$$

where a is a real constant, and f is a nonlinear continuous function, $m, n \in N_0$. Let $\Omega = N_0 \times N_0 \backslash N_0 \times N_1$. As in Section 1.2, for a given function $\varphi(i, j)$ defined on Ω, it is easy to construct a double sequence $\{x_{i,j}\}$ that equals $\varphi(i, j)$ on Ω and satisfies system (5.1) for $i, j = 0, 1, 2, \ldots$. Such double sequence is a solution of system (5.1) and is unique.

One can see that system (5.1) can be regarded as a discrete analog of the partial differential system

$$\frac{\partial v}{\partial x} + a\frac{\partial v}{\partial y} + av = f[(1+a)v]. \tag{5.2}$$

In fact, system (5.2) is a convection equation with a forced term, which is quite classical in physics. Therefore, qualitative properties of system (5.1) should provide some useful information for analyzing this companion partial differential system.

5.2.1. Spatial period orbit

First, we introduce a basic definition.

Definition 5.1. Let $V \subseteq R^3$, let V_0 be a nonempty subset of V, and take $I \subseteq V_0$, and $I \subset R$. Assume that $f : I \to I$ is a continuous map, with $f^k(x) = f(f^{k-1}(x))$ and $f^0(x) = x$. Then, f is said to be a continuous self-map in I if $f \in C^0(I, I)$ and $f(I) \subset I$. Also, x_0 is said to be a spatial periodic point of period k if $x_0 \in I$ such that

$$f^k(x_0) = x_0, \tag{5.3}$$

and $x_0 \neq f^s(x_0)$ for $1 \leq s < k$, where k is called the prime period of x_0. Moreover, the sequence

$$x_0, x_1, x_2, \dots, x_k, x_0, x_1, \dots, \tag{5.4}$$

is called a spatial period orbit of $f(x)$ with period k.

Theorem 5.2. *For any given sequence of nonzero real functions*

$$x_{mn} + ax_{mn}, x_{m+1,n} + ax_{m,n+1}, \dots, x_{m+(k-1),n} + ax_{m,n+(k-1)}, \tag{5.5}$$

if they satisfy

$$x_{m+i,n} + ax_{m,n+i} \neq x_{m+j,n} + ax_{m,n+j}, \quad i \neq j,\ i, j = 1, 2, \dots, k, \tag{5.6}$$

for $m, n \in N_0$, then the map

$$f(x+y) = a_1(x+y)^k + a_2(x+y)^{k-1} + \cdots + a_k(x+y) \tag{5.7}$$

has a periodic point $x_{mn} + ax_{mn}$ *with prime period* k*, where* $a_i = \triangle^{(i)}/\triangle$*,* $i = 1,2,\ldots,k$*, in which*

$$\triangle = \begin{vmatrix} r_0^k & r_0^{k-1} & \cdots & r_0 \\ r_1^k & r_1^{k-1} & \cdots & r_1 \\ \cdots & \cdots & \cdots & \cdots \\ r_{k-1}^k & r_{k-1}^{k-1} & \cdots & r_{k-1} \end{vmatrix}, \tag{5.8}$$

where, for simplicity, let $x_{m+i,n} + ax_{m,n+i} = r_i(m,n) = r_i$*,* $i = 0,1,2,\ldots,k-1$*, and determinant* $\triangle^{(i)}$ *is obtained from* $\triangle$ *by replacing the ith column of* $\triangle$ *with the following vector:*

$$(r_1, r_2, \ldots, r_{k-1}, r_0)^T, \quad i = 1,2,\ldots,k. \tag{5.9}$$

Proof. Given a nonzero real sequence (5.5) satisfying (5.6) for $m,n \in N_0$, suppose that the map

$$f(x+y) = a_1(x+y)^k + a_2(x+y)^{k-1} + \cdots + a_k(x+y) \tag{5.10}$$

satisfies

$$\begin{aligned} f\left(x_{mn} + ax_{mn}\right) &= x_{m+1,n} + ax_{m,n+1}, \\ f\left(x_{m+1,n} + ax_{m,n+1}\right) &= x_{m+2,n} + ax_{m,n+2}, \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ f\left(x_{m+(k-1),n} + ax_{m,n+(k-1)}\right) &= x_{mn} + ax_{mn}. \end{aligned} \tag{5.11}$$

By our simplistic notations, (5.11) is equivalent to the following system:

$$\begin{aligned} a_1 r_0^k + a_2 r_0^{k-1} + \cdots + a_k r_0 &= r_1, \\ a_1 r_1^k + a_2 r_1^{k-1} + \cdots + a_k r_1 &= r_2, \\ \vdots \qquad \vdots \qquad \vdots & \\ a_1 r_{k-2}^k + a_2 r_{k-2}^{k-1} + \cdots + a_k r_{k-2} &= r_{k-1}, \\ a_1 r_{k-1}^k + a_2 r_{k-1}^{k-1} + \cdots + a_k r_{k-1} &= r_0, \end{aligned} \tag{5.12}$$

which determines the unknown a_i, $i = 1,2,\ldots,k$.

It is easy to check that the determinant of the coefficients of system (5.12) is the kth Vandermonde determinant, and

$$\triangle = \begin{vmatrix} r_0^k & r_0^{k-1} & \cdots & r_0 \\ r_1^k & r_1^{k-1} & \cdots & r_1 \\ \cdots & \cdots & \cdots & \cdots \\ r_{k-1}^k & r_{k-1}^{k-1} & \cdots & r_{k-1} \end{vmatrix} = (-1)^{k(k-1)/2}\left(\prod_{i=0}^{k} r_i\right)\left[\prod_{k-1\geq i>j\geq 1}(r_i - r_j)\right]. \tag{5.13}$$

Since (5.6) holds, one has $\Delta \neq 0$, so that there exists a unique solution of system (5.12) given by

$$a_i^* = \frac{\Delta^{(i)}}{\Delta}, \quad i = 1, 2, \ldots, k. \tag{5.14}$$

Now, substituting $a_i = a_i^*$ into (5.10), $i = 1, 2, \ldots, k$, one has

$$f(x+y) = a_1^*(x+y)^k + a_2^*(x+y)^{k-1} + \cdots + a_k^*(x+y). \tag{5.15}$$

It is easy to see that the function f in (5.15) is continuous and

$$f^k(x_{mn} + ax_{mn}) = f^k(r_0) = r_0 = x_{mn} + ax_{mn}, \tag{5.16}$$

but

$$x_{mn} + ax_{mn} \neq f^s(x_{mn} + ax_{mn}) \quad \text{for } 1 \leq s < k. \tag{5.17}$$

Hence, the function f in (5.15) is a continuous map with a spatial periodic point $x_{mn} + ax_{mn}$ of period k. □

5.2.2. Spatial chaos

To consider chaos of (5.1), we introduce the definition of chaos in the sense of Li and Yorke.

Consider the following system:

$$x_{n+1} = F(x_n), \quad n \geq 0, \tag{5.18}$$

where $F : X \to X$ is a map and (X, d) is a metric space.

Definition 5.3. Let (X,d) be a compact metric space and let $F : X \to X$ be a continuous map. A subset S of X is called a scrambled set of F if, for any two different points $x, y \in S$,

(i) $\liminf_{n\to\infty} d(F^n(x), F^n(y)) = 0$;
(ii) $\limsup_{n\to\infty} d(F^n(x), F^n(y)) > 0$.

F is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set S of F.

For (5.1), the following result for the 1D dynamic system will be used.

Theorem 5.4 (Li-Yorke theorem). *Let $I \subset \mathbf{R}$ be an interval and let $f : I \to I$ be a continuous map. Assume that there is a point $a \in I$ satisfying*

$$f^3(a) \le a < f(a) < f^2(a) \quad \text{or} \quad f^3(a) \ge a > f(a) > f^2(a). \tag{5.19}$$

Then

(1) *for every $k = 1,2,\dots$, there is a k-periodic point of f;*
(2) *there is an uncountable set $S \subset I$, containing no periodic points such that*
  (A) *for every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0, \qquad \liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0, \tag{5.20}$$

  (B) *for every $p \in S$ and periodic point $q \in I$ with $p \neq q$,*

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0. \tag{5.21}$$

From Definition 5.3 and the above Li-Yorke Theorem, the 1D dynamical system $x_{i+1} = f(x_i)$ is chaotic in the sense of Li-Yorke if (5.19) holds.

Remark 5.5. It is known, (A) implies (B).

The following is the definition of chaos in the sense of Li-Yorke for system (5.1).

Definition 5.6. Let $V \subseteq R^3$, let V_0 be a nonempty subset of V, and take $I \subseteq V_0$, and $I \subset R$. Then, f is said to be chaotic on V_0 if it is chaotic on I, and f is said to be chaotic on V if it is chaotic on V_0, both in the sense of Li and Yorke.

Theorem 5.7. *Let $V \subseteq R^3$, let V_0 be a nonempty subset of V, and let $I \subset R$ be an interval in V_0. Denote*

$$r_i = r_i(m,n) = x_{m+i,n} + a x_{m,n+i}, \quad i = 0,1,2, \tag{5.22}$$

and assume the following conditions:

(i) $r_i(m,n) \neq 0$, $i = 0,1,2$, *and* $r_i(m,n) \neq r_j(m,n)$ *if* $i \neq j$ *for all* $m,n \in N_0$, $i,j = 0,1,2$;

(ii) $r_0(m,n) < r_1(m,n) < r_2(m,n)$ *or* $r_0(m,n) > r_1(m,n) > r_2(m,n)$ *for all* $m,n \in N_0$;

(iii) *let*

$$f^*(x+y) = a_1^*(x+y)^3 + a_2^*(x+y)^2 + a_3^*(x+y), \tag{5.23}$$

where $a_i^* = D^{(i)}/D$, $i = 1,2,3$,

$$D = \begin{vmatrix} r_0^3 & r_0^2 & r_0 \\ r_1^3 & r_1^2 & r_1 \\ r_2^3 & r_2^2 & r_2 \end{vmatrix}, \tag{5.24}$$

and the determinant $D^{(i)}$ *is obtained from* D *by replacing the ith column of* D *with the following vector:*

$$\left(x_{m+1,n} + ax_{m,n+1}, x_{m+2,n} + ax_{m,n+2}, x_{mn} + ax_{mn}\right)^T, \quad i = 1,2,3; \tag{5.25}$$

(iv) $f^*(I) \subset I$.

Then, system

$$x_{m+1,n} + ax_{m,n+1} = f^*((1+a)x_{mn}) \tag{5.26}$$

is chaotic on V *in the sense of Li and Yorke.*

Proof. Without loss of generality, consider system

$$x_{s+1,n} + ax_{m,t+1} = f^*(x_{sn} + ax_{mt}), \tag{5.27}$$

where $s,t,m,n \in N_0$. Then, one can compute and obtain the above $r_i(m,n)$, $i = 0,1,2$, where $r_0(m,n) = x_{mn} + ax_{mn}$. Next, by Theorem 5.2, we obtain

$$x_{s+1,n} + ax_{m,t+1} = a_1^*\left(x_{sn} + ax_{mt}\right)^3 + a_2^*\left(x_{sn} + ax_{mt}\right)^2 + a_3^*\left(x_{sn} + ax_{mt}\right). \tag{5.28}$$

Since $s,t,m,n \in N_0$, letting $s = m$ and $t = n$ gives

$$\begin{aligned} x_{m+1,n} + ax_{m,n+1} = {} & a_1^*\left(x_{mn} + ax_{mn}\right)^3 + a_2^*\left(x_{mn} + ax_{mn}\right)^2 \\ & + a_3^*\left(x_{mn} + ax_{mn}\right) = f^*[(1+a)x_{mn}]. \end{aligned} \tag{5.29}$$

Note that

$$\begin{gathered} f^{*3}\left(x_{mn} + ax_{mn}\right) = f^{*3}\left(r_0(m,n)\right) = r_0(m,n) = x_{mn} + ax_{mn}, \\ f^{*i}\left(r_0(m,n)\right) \neq r_0(m,n), \quad i = 1,2. \end{gathered} \tag{5.30}$$

Therefore, the map f^* has a periodic point $r_0(m,n)$ of period 3.

Since $r_i(m,n)$ are distinct points and $r_i(m,n) \neq 0$ for $m,n \in N_0$, $i = 0,1,2$, it follows immediately from Theorem 5.2 that $D \neq 0$, that is, a_i^* are uniquely determined. Therefore, $f^*(x_{m,n} + ax_{mn}) = f^*[(1+a)x_{mn}]$ is uniquely determined. In addition, since $r_i(m,n) \in I$ and $f^*(I) \subset I$, one concludes that f^* is a continuous self-map in $I \subset V_0$.

On the other hand, it follows from (ii) that

$$\begin{aligned} f^{*3}(r_0(m,n)) &= r_0(m,n) < r_1(m,n) \\ &= f^*(r_0(m,n)) < f^{*2}(r_0(m,n)) \\ &= r_2(m,n) \end{aligned} \tag{5.31}$$

for $m,n \in N_0$.

Thus, by Li-Yorke Theorem, system (5.26) is chaotic on I, from Definition 5.6, which implies that system (5.26) is chaotic on V, in the sense of Li and Yorke. □

5.3. Method of infinite-dimensional discrete dynamical systems

In this section, the following 2D discrete systems are studied:

$$x_{m+1,n} = f(x_{m,n}, x_{m,n+1}), \tag{5.32}$$

where $m,n \in N_0$ and $f : R^2 \to R$ is a function.

System (5.32) includes the following equations as special cases:

$$x_{m+1,n} = \mu x_{m,n}(1 - x_{m,n}), \tag{5.33}$$

$$x_{m+1,n} = 1 - \mu x_{m,n}^2, \tag{5.34}$$

$$x_{m+1,n} = \mu x_{m,n}(1 - x_{m,n+1}), \tag{5.35}$$

$$ax_{m+1,n} + bx_{m,n} + cx_{m,n+1} = 0. \tag{5.36}$$

Systems (5.33)–(5.35) are regular 2D discrete logistic systems in different forms.

Let n_0 be a fixed integer. If $n \equiv n_0$, then systems (5.33) and (5.34) become

$$\begin{aligned} x_{m+1,n_0} &= \mu x_{m,n_0}(1 - x_{m,n_0}), \\ x_{m+1,n_0} &= 1 - \mu x_{m,n_0}^2. \end{aligned} \tag{5.37}$$

Systems (5.37) are the standard 1D logistic systems.

Hence, system (5.32) is quite general.

Let $\Omega = \{(0,n) \mid n \in N_0\} = \{(0,0),(0,1),\dots,(0,n),\dots\}$. For any given sequence $\phi = \{\phi_{m,n}\}$ defined on Ω, it is easy to construct by induction a double-indexed sequence $x = \{x_{m,n}\}_{m,n=0}^{\infty}$ that equals the initial condition on Ω and satisfies (5.32) on $N_1 \times N_0$, which is said to be a solution of system (5.32) with the initial condition ϕ.

In order to introduce Devaney's definition of chaos for discrete dynamical systems, several preliminary concepts are first presented.

Definition 5.8. Let (X,d) be a metric space, and let $g : X \to X$ be a map on (X,d) with $x_0 \in X$. The (positive or forward) orbit $O(x_0)$ of the point x_0 is defined to the following set of points:

$$O(x_0) = \{g^n(x_0)\}_{n=0}^{\infty} = \{x_0, g(x_0), g^2(x_0), g^3(x_0), \dots\}, \tag{5.38}$$

where $g^0(x_0) = x_0$ and $g^{n+1} = g(g^n(x_0))$ for all $n \in N_0$.

Let $x_n = g^n(x_0)$. Then, the orbit $O(x_0)$ of the point $x_0 \in X$ is a 1D sequence,

$$O(x_0) = \{x_n\}_{n=0}^{\infty} = \{x_0, x_1, x_2, \dots\}. \tag{5.39}$$

Obviously, $O(x_0)$ is a solution of the 1D system $x_{n+1} = g(x_n)$, $n \in N_0$.

Let $x \in X$ and let ε be a positive number. Then, an ε-open ball $B_\varepsilon(x)$ at x is defined as $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$. A subset U of X is open if for any $x \in U$ there exists a $\delta > 0$ such that $B_\delta(x) \subseteq U$.

Let $x \in X$ and let G be an open subset of X. If $x \in G$, then G is called a neighborhood of the point x. Let $U \subseteq V$ be two subsets of X. If, for any $x \in V$ and any small $\varepsilon > 0$, $B_\varepsilon(x) \cap U \neq \emptyset$, where $\emptyset$ denotes the empty set, then the set U is said to be dense in V. Especially, if $V = X$, then U is a dense subset of X.

The definition of chaos in the sense of Devaney contains three important ingredients, that is, dense periodic points, transitivity, and sensitive dependence on initial conditions, defined as follows.

Definition 5.9. Let $g : X \to X$ be a continuous map on a metric space (X,d) and $x \in X$. If there exists a positive integer n such that $g^n(x) = x$, then x is called a periodic point of g and n is called a period of x. If $g^n(x) = x$ and $g^k(x) \neq x$ for all $k = 1, 2, \dots, n-1$, then x is called a primitive n-periodic point and n is called the prime period of x. In particular, if $n = 1$, then x is called a fixed point of the map g.

If, for any point $a \in X$ and any neighborhood U of the point a, there exist a point $x \in U$ and an integer $n \in N_1$ such that $g^n(x) = x$, then it is said that system (5.32) has a dense set of periodic points (in X).

If, for any two nonempty open subsets U and V of X, there is an integer $k > 0$ such that $g^k(U) \cap V \neq \emptyset$, then the map g is said to be (topologically) transitive on X.

If there is a $\delta > 0$, called a sensitivity constant such that, for each point $x \in X$ and each neighborhood G of x, there exist a point $y \in G$ and a positive integer n such that $d(g^n(x), g^n(y)) > \delta$, then it is said that system (5.32) has sensitive dependence on initial conditions.

The above sensitivity condition captures the idea that in chaotic systems a tiny difference in initial value eventually leads to a large-scale divergence.

Definition 5.10. Let $F : X \to X$ be a continuous map on a metric space (X, d). The map F is said to be chaotic in the sense of Devaney on X if

(1) F is transitive on X;
(2) the set of periodic points of F is dense in X;
(3) F has sensitive dependence on initial conditions.

It was pointed out that if F is continuous, then condition (3) is implied by other two conditions (1) and (2), which shows that the sensitive dependence is redundant in Devaney's definition of chaos for a continuous map F.

Remark 5.11. Huang and Ye [71] point out that under some conditions, chaos in the sense of Devaney is stronger than that in the sense of Li-Yorke.

Let R^∞ be a set of all 1D real sequences, that is,

$$R^\infty = \{\{a_n\}_{n=0}^{\infty} = (a_0, a_1, \dots, a_n, \dots) \mid a_n \in R,\ n \in N_0\}. \tag{5.40}$$

Obviously, different metrics can be defined on R^∞. For example, for any two sequences, $x_1 = \{x_{1,n}\}_{n=0}^{\infty}, x_2 = \{x_{2,n}\}_{n=0}^{\infty} \in R^\infty$, one may define

$$d_1(x_1, x_2) = \sum_{n=0}^{\infty} \frac{|x_{1,n} - x_{2,n}|}{2^n}, \tag{5.41}$$

or

$$d_2(x_1, x_2) = \sup\{|x_{1,n} - x_{2,n}| \mid n = 0, 1, 2, \dots\}. \tag{5.42}$$

Then, it is easy to prove that d_1 and d_2 define two metrics on two subsets of R^∞. For simplicity of notations, use (R^∞, d_1) and (R^∞, d_2) to denote the two metric spaces defined by these two metrics, respectively. Note that both (R^∞, d_1) and (R^∞, d_2) are complete metric spaces.

In the following, for convenience, (R^∞, d) is used to denote a metric space with any metric d including d_1 and d_2 defined above.

Let I be a subset of R and denote

$$I^\infty = \{\{a_n\}_{n=0}^{\infty} = (a_0, a_1, \dots, a_n, \dots) \mid a_n \in I,\ n \in N_0\}. \tag{5.43}$$

It is obvious that (I^∞, d) is also a metric space and I^∞ is a metric subspace of R^∞.

Let $f : I \times I \to I$ be a function and let $x = \{x_{m,n}\}_{m,n=0}^{\infty}$ be any solution of system (5.32) with the initial condition $\phi = \phi_0 = \{\phi_n = \phi_{0,n}\}_{n=0}^{\infty}$, where $\phi_n \in I$ for all $n \in N_0$, and denote

$$x_m = \{x_{m,n}\}_{n=0}^{\infty} = (x_{m,0}, x_{m,1}, x_{m,2}, \dots) \quad \text{for any } m = 0, 1, 2, \dots. \tag{5.44}$$

Note that $x_{m,n} \in I$ for all $(m,n) \in N_0 \times N_0$, and for any $m \in N_0$, x_m is a 1D sequence, $x_0 = \phi_0$, $x_m \in I^\infty$, and for all $m \in N_0$, denote

$$x_{m+1} = (x_{m+1,0}, x_{m+1,1}, \dots) = (f(x_{m,0}, x_{m,1}), f(x_{m,1}, x_{m,2}), \dots) = F(x_m). \tag{5.45}$$

System (5.32) is equivalent to the following system defined in the metric space (I^∞, d):

$$x_{m+1} = F(x_m), \quad m \in N_0. \tag{5.46}$$

The map F defined in (5.45) is said to be induced by system (5.32) and (f, F) is a pair of maps associated with the two systems.

Obviously, a double-indexed sequence $\{x_{m,n}\}_{m,n=0}^\infty$ is a solution of system (5.32) if and only if the sequence $\{x_m\}_{m=0}^\infty$ is a solution of system (5.46), where $x_m = \{x_{m,n}\}_{n=0}^\infty$, $m \in N_0$.

Definition 5.12. Let I be a subset of R, let $f : I \times I \to I$ be a function, and let $F : I^\infty \to I^\infty$ be a map on the metric space (I^∞, d) induced by system (5.32). If system (5.46) is chaotic on I^∞ in the sense of Devaney (or Li-Yorke), then system (5.32) is said to be chaotic on I^∞ in the sense of Devaney (or Li-Yorke).

In the following, an example is given to illustrate that system (5.32) indeed is chaotic in the sense of Devaney under the given conditions.

Consider a 2D discrete system of the form

$$x_{m+1,n} = f(x_{m,n}, x_{m,n+1}), \quad m, n = 0, 1, 2, \dots, \tag{5.47}$$

where $f : I \times I \to I$ is a function defined $f(x,y) = \langle x + y \rangle$ for any $x, y \in I$, in which $\langle a \rangle$ denotes the decimal part of the real number a, and $I = [0, 1)$.

Let $a \in I = [0,1)$, $b \in [0,1]$, and

$$\lceil a, b \rceil = \begin{cases} (a,b), & 0 \le a < b \le 1, \\ (a,1) \cup [0,b), & 0 \le b \le a < 1. \end{cases} \tag{5.48}$$

Denote a set Δ by

$$\Delta = \{\lceil a, b \rceil \mid a \in I,\ b \in [0,1]\} \cup \{[0,1)\}. \tag{5.49}$$

Lemma 5.13. *Let* $\lceil a_1, b_1 \rceil, \lceil a_2, b_2 \rceil \in \Delta$. *Then,* $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) \in \Delta$ *and*

$$|f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil)| \geq \min\{1, |\lceil a_1, b_1 \rceil| + |\lceil a_2, b_2 \rceil|\}, \tag{5.50}$$

where $|\lceil a, b \rceil|$ *denotes the Lebesgue measure for any* $\lceil a, b \rceil \in \Delta$.

Proof. (1) If $\lceil a_1, b_1 \rceil = [0,1)$ or $\lceil a_2, b_2 \rceil = [0,1)$, then the conclusions of Lemma 5.13 hold obviously.

(2) If $0 \leq a_1 < b_1 \leq 1$ and $0 \leq a_2 < b_2 \leq 1$, then $\lceil a_1, b_1 \rceil = (a_1, b_1)$ and $\lceil a_2, b_2 \rceil = (a_2, b_2)$.

If $a_1 + a_2, b_1 + b_2 \in [0,1]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1 + a_2, b_1 + b_2) \in \Delta$. Hence (5.50) holds.

If $a_1 + a_2 \in [0,1]$ and $b_1 + b_2 \in [1,2]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1 + a_2, 1) \cup [0, b_1 + b_2 - 1) \in \Delta$. Hence (5.50) holds.

If $a_1+a_2, b_1+b_2 \in [1,2]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1+a_2-1, b_1+b_2-1) \in \Delta$. Hence (5.50) holds.

(3) If $0 \leq b_1 \leq a_1 < 1$ and $0 \leq a_2 < b_2 \leq 1$, then $\lceil a_1, b_1 \rceil = (a_1, 1) \cup [0, b_1)$ and $\lceil a_2, b_2 \rceil = (a_2, b_2)$.

If $b_1 + a_2, b_1 + b_2, a_1 + a_2, a_1 + b_2 \in [0,1)$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1 + a_2, 1) \cup [0, b_2) \cup (a_2, b_1 + b_2) = (a_1 + a_2, 1) \cup [0, b_1 + b_2) \in \Delta$. Hence (5.50) holds.

If $b_1 + a_2, b_1 + b_2, a_1 + a_2 \in [0,1)$, and $a_1 + b_2 \in [1,2]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1 + a_2, 1) \cup [0, b_2) \cup (a_2, b_1 + b_2) = (a_1 + a_2, 1) \cup [0, b_1 + b_2) \in \Delta$. Hence (5.50) holds.

If $b_1 + a_2, b_1 + b_2 \in [0,1)$ and $a_1 + a_2, a_1 + b_2 \in [1,2]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1 + a_2 - 1, b_2) \cup (a_2, b_1 + b_2) = (a_1 + a_2 - 1, b_1 + b_2) \in \Delta$. Hence (5.50) holds.

If $b_1 + a_2, a_1 + a_2 \in [0,1)$ and $b_1 + b_2, a_1 + b_2 \in [1,2]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1 + a_2, 1) \cup [0, b_2) \cup (a_2, 1) \cup [0, b_1 + b_2 - 1) = (a_2, 1) \cup [0, b_2) \in \Delta$. Hence (5.50) holds.

If $b_1 + a_2 \in [0,1)$ and $b_1 + b_2, a_1 + a_2, a_1 + b_2 \in [1,2]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1+a_2-1, b_2) \cup (a_2, 1) \cup [0, b_1+b_2-1) = (a_1+a_2-1, 1) \cup [0, b_1+b_2-1) \in \Delta$. Hence (5.50) holds.

If $b_1 + a_2, b_1 + b_2, a_1 + a_2, a_1 + b_2 \in [1,2)$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1 + a_2 - 1, b_2) \cup [a_2, 1) \cup [0, b_1 + b_2 - 1) = (a_1 + a_2 - 1, 1) \cup [0, b_1 + b_2 - 1) \in \Delta$. Hence (5.50) holds.

(4) If $0 \leq b_2 \leq a_2 < 1$ and $0 \leq a_1 < b_1 \leq 1$, then $\lceil a_2, b_2 \rceil = (a_2, 1) \cup [0, b_2)$ and $\lceil a_1, b_1 \rceil = (a_1, b_1)$. Similar to the proof in (3), the conclusions of Lemma 5.13 hold.

(5) If $0 \leq b_1 \leq a_1 < 1$ and $0 \leq b_2 \leq a_2 \leq 1$, then $\lceil a_1, b_1 \rceil = (a_1, 1) \cup [0, b_1)$ and $\lceil a_2, b_2 \rceil = (a_2, 1) \cup [0, b_2)$.

If $a_1 + a_2, b_1 + b_2 \in [0,1]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = [0,1) \in \Delta$. Hence (5.50) holds.

If $b_1 + b_2 \in [0,1]$ and $a_1 + a_2 \in [1,2]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = (a_1 + a_2 - 1, 1) \cup [a_1, 1) \cup [0, b_2) \cup (a_2, 1) \cup [0, b_1) \cup [0, b_1+b_2) = (a_1+a_2-1, 1) \cup [0, b_1+b_2) \in \Delta$. Hence (5.50) holds.

If $b_1 + b_2, a_1 + a_2 \in [1,2]$, then $f(\lceil a_1, b_1 \rceil \times \lceil a_2, b_2 \rceil) = [0,1) \in \Delta$. Hence (5.50) holds.

From the proof for cases (1)–(5), one can see that the conclusions of Lemma 5.13 hold. The proof is thus completed. □

Let $F : I^\infty \to I^\infty$ be a map in (I^∞, d) induced by system (5.47). Then, from Lemma 5.13, the following result holds.

Corollary 5.14. *Let $\varepsilon > 0$ be a constant and $(a_i, b_i) \subset I = [0,1)$, with $b_i - a_i = \varepsilon_i \geq \varepsilon$ for all $i \in N_0$. Then there exists an integer $n > 0$ such that*

$$F^n\left(\prod_{i=0}^{\infty}(a_i, b_i)\right) = F^n((a_0, b_0) \times (a_1, b_1) \times \cdots) = I^\infty. \tag{5.51}$$

Proof. From the given conditions, $(a_i, b_i), (a_{i+1}, b_{i+1}) \in \Delta$ for all $i \in N_0$. Hence, from Lemma 5.15, one has

$$\begin{aligned} F^n\left(\prod_{i=0}^{\infty}(a_i, b_i)\right) &= \left(f((a_0, b_0) \times (a_1, b_1)), f((a_1, b_1) \times (a_2, b_2)), \ldots\right) \\ &= \prod_{i=0}^{\infty}\lceil a_i^1, b_i^1 \rceil, \end{aligned} \tag{5.52}$$

where $\lceil a_i^1, b_i^1 \rceil = f((a_i, b_i) \times (a_{i+1}, b_{i+1}))$ for all $i \in N_0$. Thus, in view of Corollary 5.14 and the given conditions, $|\lceil a_i^1, b_i^1 \rceil| \geq \min\{1, 2\varepsilon\}$ for all $i \in N_0$. By the iterative method, it can be verified that there exists an integer $n > 0$ such that (5.51) holds. The proof is completed. □

Now, one can prove that F induced by (5.47) is chaotic on (I^∞, d_1) in the sense of Devaney, where d_1 is defined by (5.41).

First, it is to prove that F is transitive on I^∞.

Let U and V be two nonempty open subsets of I^∞. Since d_1 is a metric of I, there exist a number $\varepsilon > 0$ and a set $\prod_{i=0}^{\infty}(a_i, b_i) \subseteq U$ with $|b_i - a_i| > \varepsilon$. Hence, from Corollary 5.14, there exists an integer $k > 0$ such that $F^k(U) = I^\infty$. Therefore, $F^k(U) \cap V = V \neq \emptyset$. Thus, the map F is transitive in I^∞.

Second, it is to prove that F has a dense set of periodic points.

Lemma 5.15. *For any integer $n \in N_0$, let A_n be a set of all solutions of the following equations:*

$$F(x_0) = x_1, F(x_1) = x_2, \ldots, F(x_{n-1}) = x_n, F(x_n) = x_0. \tag{5.53}$$

Then, the set $A = \bigcup_{n=0}^{\infty} A_n$ of periodic points of F is dense in I^∞ on the metric space (I^∞, d_1), where d_1 is defined by (5.41).

Proof. Since (f, F) is pair of maps, (5.53) is equivalent to the following equations:

$$\begin{aligned} x_{1,0} &= \langle x_{0,0} + x_{0,1} \rangle, \dots, x_{1,k} = \langle x_{0,k} + x_{0,k+1} \rangle, \dots, \quad k = 0, 1, \dots, \\ x_{2,0} &= \langle x_{1,0} + x_{1,1} \rangle, \dots, x_{2,k} = \langle x_{1,k} + x_{1,k+1} \rangle, \dots, \quad k = 0, 1, \dots, \\ &\cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ x_{0,0} &= \langle x_{n,0} + x_{n,1} \rangle, \dots, x_{0,k} = \langle x_{n,k} + x_{n,k+1} \rangle, \dots, \quad k = 0, 1, \dots, \end{aligned} \tag{5.54}$$

which implies that

$$x_{0,k} = \left\langle \sum_{m=0}^{n+1} C_{n+1}^{m} x_{0,k+m} \right\rangle, \quad k = 0, 1, \dots. \tag{5.55}$$

Let $a = \{a_j\}_{j=0}^{\infty}$ be any point of I^{∞}, let ε be any small positive number, and let $B_{\varepsilon}(a) \subseteq I^{\infty}$ be any open ball at the center a. Define a sequence of sets as follows:

$$V_M = \left\{ b \in I^{\infty} \mid |b_0 - a_0| < \frac{\varepsilon}{3}, \dots, |b_M - a_M| < \frac{\varepsilon}{3},\ b_j \in [0, 1),\ j = M+1, M+2, \dots \right\}, \tag{5.56}$$

where $M \in N_0$. Then, from the definition of the metric d_1 in (5.41), one can see that there exists an integer $M_0 > 0$ such that for any $b = \{b_j\}_{j=0}^{\infty} \in V_{M_0}$, $d_1(a, b) < \varepsilon$, that is, $V_{M_0} \subseteq B_{\varepsilon}(a) \subseteq I^{\infty}$.

Take a sufficiently large integer $p > 0$ such that for any point $y \in I = [0, 1)$ there exists an integer $q \in \{0, 1, \dots, p-1\}$ satisfying $|q/p - y| < \varepsilon/6$. Then, there exist integers $q_0, q_1, \dots, q_{M_0} \in \{0, 1, \dots, p-1\}$ such that $|q_j/p - a_j| < \varepsilon/3$ for $j = 0, 1, \dots, M_0$.

Let $n = p$. In view of (5.55), it is obvious that there exist integers $q_j \in \{0, 1, \dots, p-1\}$ such that $q_j/p \in I$ for all $j \in \{M_0 + 1, M_0 + 2, \dots\}$ and (5.55) holds for the point $x_0 = \{x_{0,j} = q_j/p\}_{j=0}^{\infty}$. It is easy to verify that the point $x_0 = \{x_{0,j} = q_j/p\}_{j=0}^{\infty} \in I^{\infty}$ is a periodic point of F with period $p + 1$ and $x_0 \in B_{\varepsilon}(a)$. Hence, the set $A = \bigcup_{n=0}^{\infty} A_n$ of periodic points of F is dense in I^{∞}. The proof is thus completed. □

In view of Lemma 5.15 and its proof, it is clear that F has a dense set of periodic points.

Third, it is to prove that F has sensitive dependence on initial conditions.

Let $\delta = 0.1$, let $a = \{a_i\}_{i=0}^{\infty} \in I^{\infty}$ be any point, and let U be any neighborhood of a. In view of Corollary 5.14 and the proof of Lemma 5.13, it is obvious that there exist a constant $\varepsilon_0 > 0$, a point $b \in B_{\varepsilon_0}(a)$ with $b \neq a$, and an integer $n \in N_1$, such that $B_{\varepsilon_0}(a) \subseteq U$ and $d_1(F^n(b), F^n(a)) > \delta$, that is, F has sensitive dependence on initial conditions.

Therefore, F induced by system (5.47) is chaotic on (I^{∞}, d_1) in the sense of Devaney. Then system (5.47) is chaotic in the sense of Devaney.

5.4. Criteria of chaos in complete metric spaces

Consider the following discrete dynamical system:

$$x_{n+1} = F(x_n), \quad n \geq 0, \tag{5.57}$$

where $F : X \to X$ is a map and (X, d) is a metric space.

Definition 5.10 about chaos is for the space X. Since chaos of F often appears on a subset of X, it is necessary to give a corresponding definition of chaos of a map on a subset. Let V be a subset of a metric space (X, d). A continuous map $F : V \to V$ is said to be chaotic on V in the sense of Devaney if F satisfies properties (1) and (2) in Definition 5.10 on V.

At first, we give some definitions and prepare several lemmas. For the convenience of the following discussion, we first introduce some notations.

Let (X, d) be a metric space, $x \in X$, and let A, B be subsets of X. The boundary of A, denoted by ∂A, is the set of all $x \in X$ such that each neighborhood of x intersects both A and $X \setminus A$; the distance between the point x and the set A is denoted by

$$d(x, A) = \inf \{d(x, y) \mid y \in A\}; \tag{5.58}$$

the distance between two sets A and B, respectively, is denoted by

$$d(A, B) = \inf \{d(x, y) \mid x \in A, \ y \in B\}; \tag{5.59}$$

the maximal distance between two points in A and B is denoted by

$$d_s(A, B) = \sup \{d(x, y) \mid x \in A, \ y \in B\}; \tag{5.60}$$

and the diameter of the set A is denoted by

$$d(A) = \sup \{d(x, y) \mid x, y \in A\}. \tag{5.61}$$

Definition 5.16. Let (X, d) be a metric space and let $F : X \to X$ be a map. A point $z \in X$ is called an expanding (or repelling) fixed point (or a repeller) of F in $\overline{B}_r(z)$ for some constant $r > 0$ if $F(z) = z$ and there exists a constant $\lambda > 1$ such that

$$d(F(x), F(y)) \geq \lambda d(x, y) \quad \forall x, y \in \overline{B}_r(z), \tag{5.62}$$

where $\overline{B}_r(z)$ is the closed ball centered at z, that is, $\overline{B}_r(z) = \{x \in X \mid d(x, z) \leq r\}$. The constant λ is called an expanding coefficient of F in $\overline{B}_r(z)$.

Definition 5.17. Assume that z is an expanding fixed point of F in $\overline{B}_r(z)$ for some $r > 0$. Then z is said to be a snap-back repeller of F if there exists a point $x_0 \in B_r(z)$ with $x_0 \neq z$ and $F^m(x_0) = z$ for some positive integer m, where $B_r(z)$ is the open ball centered at z.

Definition 5.18. Assume that z is a snap-back repeller of F, associated with an x_0, an m, and an r as specified in Definition 5.17. Then z is said to be a nondegenerate snap-back repeller of F if there exist positive constants μ and $r_0 < r$ such that $B_{r_0}(x_0) \subset B_r(z)$ and

$$d(F^m(x), F^m(y)) \geq \mu d(x,y) \quad \forall x, y \in \overline{B}_{r_0}(x_0). \tag{5.63}$$

Next, we study the expansion of sets near an expanding fixed point. In general, a map may not expand the sets near its expanding fixed point. However, we have the following result.

Lemma 5.19. *Let (X,d) be a metric space and let $F : X \to X$ be map with an expanding fixed point z in $\overline{B}_{r^*}(z)$ for some $r^* > 0$. If F is continuous on $\overline{B}_{r^*}(z)$ and z is an interior point of $F(\overline{B}_{r^*}(z))$, then there exists a positive constant $r_0 \leq r^*$ such that for each positive constant $r \leq r_0$, $F(\overline{B}_r(z))$ is closed set, $F(B_r(z))$ is an open set, and*

$$F(\overline{B}_r(z)) \supset \overline{B}_r(z), \qquad F(B_r(z)) \supset B_r(z). \tag{5.64}$$

Proof. Suppose that $\lambda > 1$ is an expanding coefficient of F in $\overline{B}_{r^*}(z)$, then

$$d(F(x), F(y)) \geq \lambda d(x,y) \quad \forall x, y \in \overline{B}_{r^*}(z). \tag{5.65}$$

By the assumption, z is an interior point of $F(\overline{B}_{r^*}(z))$. So there exists a constant $\delta_0 > 0$ such that $B_{\delta_0}(z) \subset F(\overline{B}_{r^*}(z))$. It follows that $F^{-1}(B_{\delta_0}(z))$ is open from the continuity of F. Then there exists a positive constant $r_0 \leq r^*$ such that $B_{r_0}(z) \subset F^{-1}(B_{\delta_0}(z)) \subset B_{r^*}(z)$. It is evident that

$$F : F^{-1}(B_{\delta_0}(z)) \longrightarrow B_{\delta_0}(z) \tag{5.66}$$

is bijective and continuous. Now, we show that the inverse

$$F^{-1} : B_{\delta_0}(z) \longrightarrow F^{-1}(B_{\delta_0}(z)) \tag{5.67}$$

is continuous. If it is the case, then F is homeomorphic on $F^{-1}(B_{\delta_0}(z))$ and then $F(B_r(z))$ is open and $F(\overline{B}_r(z))$ is closed for each positive constant $r \leq r_0$. In fact, for any $x, y \epsilon B_{\delta_0}(z)$, $F^{-1}(x), F^{-1}(y) \in F^{-1}(B_{\delta_0}(z))$ and then

$$d(F^{-1}(x), F^{-1}(y)) \leq \lambda^{-1} d(x,y), \tag{5.68}$$

which implies that F^{-1} is continuous on $B_{\delta_0}(z)$.

For each positive constant $r \leq r_0$, it follows from the above discussion that $\partial F(B_r(z)) \subset F(\partial B_r(z))$ and $\overline{F(B_r(z))} \subset F(\overline{B}_r(z))$. For each $x \in \partial B_r(z)$, we have

$$d(F(x), z) \geq \lambda d(x,z) = \lambda r, \tag{5.69}$$

which implies that $d = d(z, \partial F(B_r(z))) \geq \lambda r > r$. Hence, it follows that

$$B_r(z) \subset B_d(z) \subset F(B_r(z)), \qquad \overline{B}_r(z) \subset \overline{B}_d(z) \subset \overline{F(B_r(z))} \subset F(\overline{B}_r(z)), \tag{5.70}$$

which implies that (5.64) holds. This completes the proof. □

Definition 5.20. Let (X, d) be a metric space and let $F : X \to X$ be a map.

(1) Assume that $z \in X$ is an expanding fixed point of F in $\overline{B}_r(z)$ for some constant $r > 0$. Then z is called a regular expanding fixed point of F in $\overline{B}_r(z)$ if z is an interior point of $F(B_r(z))$. Otherwise, z is called a singular expanding fixed point of F in $\overline{B}_r(z)$.

(2) Assume that z is snap-back repeller of F, associated with x_0, m, and r as specified in Definition 5.17. Then z is called a regular snap-back repeller of F if $F(B_r(z))$ is open and there exists a positive constant δ_0 such that $B_{\delta_0(x_0)} \subset B_r(z)$ and for each positive constant $\delta \leq \delta_0$, z is an interior point of $F^m(B_\delta(x_0))$. Otherwise, z is called a singular snap-back repeller of F.

Remark 5.21. (1) In (2) of Definition 5.20, the condition "$F(B_r(z))$ is open" ensures that z is a regular expanding fixed point of F in $\overline{B}_r(z)$.

(2) Suppose that z is a nondegenerate snap-back repeller of F, associated with x_0, m, r, r_0, and μ as specified in Definitions 5.17 and 5.18. If $F(B_r(z))$ is open, z is an interior point of $F^m(B_{r_0}(x_0))$, and F^m is continuous on $B_{r_0}(x_0)$, then for each positive constant $\delta \leq r_0$, z is an interior point of $F^m(B_\delta(x_0))$ by a similar argument to the proof of Lemma 5.19 and consequently, z is a regular nondegenerate snap-back repeller of F.

Next, we extend the concepts of homoclinic point and heteroclinic point and the concept of local unstable set of a repeller of a differentiable function on R, to that of a continuous map on a metric space. Before that, we first establish the following results.

Lemma 5.22. *Let (X, d) be a metric space and let $F : X \to X$ be a map with a regular expanding fixed point z in $\overline{B}_{r^*}(z)$ for some $r^* > 0$. If F is continuous on $\overline{B}_{r^*}(z)$, then there exists an open neighborhood U of z such that*

(1) *for each $x \in U$ with $x \neq z$, there exists an integer $k \geq 1$ such that $F^k(x) \notin U$;*

(2) *for each $x \in U$ with $x \neq z$, $F^{-n}(x)$ is uniquely defined in U for all $n \geq 1$, and $F^{-n}(x) \to z$ as $n \to \infty$.*

Proof. By Lemma 5.19, there exists a positive constant $r \leq r^*$ such that $F(B_r(z))$ is open and $F(B_r(z)) \supset B_r(z)$. Set $U = B_r(z)$. By Definition 5.16, there is a constant $\lambda > 1$ such that

$$d(F(x), F(y)) \geq \lambda d(x, y) \quad \forall x, y \in \overline{B}_r(z). \tag{5.71}$$

Then, for each $x \in B_r(z)$ with $x \neq z$, there exists an integer $k \geq 1$ such that $F^k(x) \notin B_r(z)$. Otherwise, there exists a point $x_0 \in B_r(z)$ with $x_0 \neq z$ such that $F^k(x_0) \in B_r(z)$ for all $k \geq 1$. It follows from (5.71) that

$$d(F^k(x_0), z) \geq \lambda d(F^{k-1}(x_0), z) \geq \lambda^k d(x_0, z) \quad \forall k \geq 1, \tag{5.72}$$

which implies that

$$\lambda^k d(x_0, z) \leq r \quad \forall k \geq 1. \tag{5.73}$$

Since $d(x_0, z) > 0$ and $\lambda > 1$, this is impossible.

Let x be any point in U. Since $F(U) \supset U$ and F is injective in U, $F^{-n}(x)$ is uniquely defined in U for all $n \geq 1$. In addition, by using the fact that F is expanding in $U = B_r(z)$, it is easily concluded that $F^{-n}(x) \to z$ as $n \to \infty$. Therefore, the proof is complete. □

Based on Lemma 5.22, we now introduce the following definitions.

Definition 5.23. Let (X, d) be a metric space and let $F : X \to X$ be a continuous map with a regular expanding fixed point $z \in X$. Let U be the maximal open neighborhood of z such that for each $x \in U$ with $x \neq z$ there exists an integer $k \geq 1$ with $F^k(x) \notin U$ and for each $x \in U$ with $x \neq z$, $F^{-n}(x)$ is uniquely defined in U with $F^{-n}(x) \to z$ as $n \to \infty$. This set U is called the local unstable set of F at z and is denoted by $W^u_{\text{loc}}(z)$.

Clearly it is possible that $W^u_{\text{loc}}(z) = X$.

Definition 5.24. Let (X, d) be a metric space and let $F : X \to X$ be a continuous map with a regular expanding fixed point $z \in X$.

(1) A point x is called homoclinic to z if $x \in W^u_{\text{loc}}(z)$, $x \neq z$, and there exists an $n \geq 1$ such that $F^n(x) = z$. The homoclinic point x, together with its backward orbit $\{F^{-j}(x)\}_{j=1}^{\infty}$ and its finite forward orbit $\{F^j(x)\}_{j=1}^{n-1}$, is called a homoclinic orbit from z.

(2) A homoclinic orbit is called nondegenerate if, for each point x_0 on the orbit, there exist positive constants r_0 and μ such that

$$d(F(x), F(y)) \geq \mu d(x, y) \quad \forall x, y \in \overline{B}_{r_0}(x_0). \tag{5.74}$$

(3) A homoclinic orbit is called regular if, for each point x_0 on the orbit, there exists a positive constant r_1 such that, for each positive constant $r \leq r_1$, $F(x_0)$ is an interior point of $F(B_r(x_0))$. Otherwise, it is called singular.

(4) A point x is called heteroclinic to z if $x \in W^u_{\text{loc}}(z)$ and there exists an $n \geq 1$ such that $F^n(x)$ lies on a different periodic orbit from z.

We next establish a fixed point theorem for an expanding continuous map in a complete metric space. Here, $F : V \to X$ is expanding in $V \subset X$ if there exists a constant $\lambda > 1$ such that

$$d(F(x), F(y)) \geq \lambda d(x, y) \quad \forall x, y \in V. \tag{5.75}$$

Lemma 5.25. *Let (X, d) be a complete metric space and let V be a nonempty closed subset of X. If $F : V \to X$ is continuous, satisfies $F(V) \supset V$, and F is expanding in V, then F has a unique fixed point in V.*

Proof. It is clear that $F : V \to F(V)$ is bijective. Consider its inverse $F^{-1} : F(V) \to V$. We first show that F^{-1} has a unique fixed point in $F(V)$. With a similar argument to that in the proof of Lemma 5.19, it is easily concluded that $F(V)$ is closed. On the other hand, from (5.75) it follows that

$$d(F^{-1}(y), F^{-1}(z)) \leq \lambda^{-1} d(y, z) \quad \forall y, z \in F(V), \tag{5.76}$$

which implies that F^{-1} is contractive on $F(V)$ since $\lambda > 1$. Hence, by the Banach contraction mapping principle in complete metric spaces, F^{-1} has a unique fixed point $y^* \in F(V)$. It is clear that y^* is also a fixed point of F in V. The uniqueness of the fixed point of F in V is easily derived from (5.75). This completes the proof. □

All the criteria of chaos obtained in this section are related to Cantor sets in metric spaces and a symbolic dynamical system, which has plentiful dynamical structures. As a matter of convenience, we introduce the concept of Cantor set in a general topological space. We also present some relevant results of symbolic dynamical systems.

Definition 5.26. Let X be a topological space and let $\wedge$ be a subset in X. Then $\wedge$ is said to be a Cantor set if it is compact, totally disconnected, and perfect. A set in X is totally disconnected if each of its connected component is a single point; a set is perfect if it is closed and every point in it is an accumulation point or a limit point of other points in the set.

Let

$$\sum_{2}^{+} := \{s = (s_0 s_1 s_2 \cdots) : s_j = 0 \text{ or } 1\} \tag{5.77}$$

and define a distance between two points $s = (s_0 s_1 s_2 \cdots)$ and $t = (t_0 t_1 t_2 \cdots)$ by

$$\rho(s, t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}. \tag{5.78}$$

For any $s, t \in \sum_2^+$, $\rho(s, t) \leq 1/2^n$ of $s_i = t_i$ for $0 \leq i \leq n$. Conversely, if $\rho(s, t) < 1/2^n$, then $s_i = t_i$ for $0 \leq i \leq n$.

Lemma 5.27. $(\sum_2^+, \rho)$ *is a complete, compact, totally disconnected, and perfect metric space.*

Proof. The completeness of $(\sum_2^+, \rho)$ can be easily proved. By Devaney [52, Part 1, Theorem 7.2], $(\sum_2^+, \rho)$ is homeomorphic to a Cantor set $\wedge_0$ in the real line R. It is well known that $\wedge_0$ is compact, totally disconnected, and perfect. Since the compactness, total disconnectedness, and perfectness are topological properties, this lemma is proved. □

The shift map $\sigma : \sum_2^+ \to \sum_2^+$ defined by $\sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 \cdots)$ is continuous. The dynamical system governed by σ is called a symbolic dynamical system and it has the following properties.

Lemma 5.28 (see [52]). (1) $\operatorname{Card}\operatorname{Per}_n(\sigma) = 2^n$,

(2) $\operatorname{Per}(\sigma)$ *is dense in* $\sum_2^+$,

(3) *there exists a dense orbit of* σ *in* $\sum_2^+$, *where* $\operatorname{Card}\operatorname{Per}_n(\sigma)$ *denotes the number of periodic points of period* n *for* σ.

It is clear that property (3) implies that σ *is transitive. Hence, this symbolic dynamical system is chaotic in the sense of Devaney.*

There is a well-known theorem in the topology theory: a topological space X is compact if and only if each collection of closed subsets of X having the finite intersection property (i.e., every finite subset has a nonempty intersection) has nonempty intersection. If X is not compact, the finite intersection property of closed subsets of X does not imply nonempty intersection in general. However, we have the following result for a complete metric space, which will be used in the proof of Theorem 5.30.

Lemma 5.29. *Let* (X, d) *be a complete metric space and let* $\{A_n\}$ *be a sequence of bounded and closed subsets of* X *which have the finite intersection property. If the diameter* $d(A_n) \to 0$ *as* $n \to \infty$, *then* $\{A_n\}$ *has a nonempty intersection, that is,*

$$\bigcap_{n\geq 1} A_n \neq \phi. \tag{5.79}$$

Furthermore, $\bigcap_{n\geq 1} A_n$ *contains only one point.*

Proof. Let

$$B_n = \bigcap_{1\leq m\leq n} A_m, \quad n \geq 1. \tag{5.80}$$

Then $\{B_n\}$ forms a nested sequence of nonempty, bounded, and closed subsets of X. Further, $B_n \subset A_n$ for all $n \geq 1$ and consequently,

$$d(B_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.81}$$

Choose a point $x_n \in B_n$ for each $n \geq 1$. Then we get a sequence $\{x_n\}$ which satisfies

$$x_m \in B_n \quad \text{for } m \geq n \tag{5.82}$$

by the nestedness of $\{B_n\}$. Then, from (5.81), we have

$$d(x_n, x_m) \leq d(B_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{5.83}$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges. Suppose $x_n \to x^*$ as $n \to \infty$. Then, $x^* \in B_n$ for all $n \geq 1$ by (5.82) and by the closeness of B_n. This implies that

$$x^* \in \bigcap_{n\geq 1} B_n \subset \bigcap_{n\geq 1} A_n \tag{5.84}$$

and consequently, (5.79) is proved. In addition, by using the condition that $d(A_n) \to 0$ as $n \to \infty$, it follows that $\bigcap_{n\geq 1} A_n$ contains only one point. Thus, the proof is complete. □

In the following, we establish two criteria of chaos generated from continuous maps in complete metric spaces and in compact subsets of metric spaces, respectively.

Theorem 5.30. *Let (x, d) be a complete metric space and let V_0, V_1 be nonempty, closed, and bounded subsets of X with $d(V_0, V_1) > 0$. If a continuous map $F : V_0 \cup V_1 \to X$ satisfies*

(1) *$F(V_j) \supset V_0 \cup V_1$ for $j = 0, 1$;*

(2) *F is expanding in V_0 and V_1, respectively, that is, there exists a constant $\lambda_0 > 1$ such that*

$$d(F(x), F(y)) \geq \lambda_0 d(x, y) \quad \forall x, y \in V_0, \ \forall x, y \in V_1; \tag{5.85}$$

(3) *there exists a constant $\mu_0 > 0$ such that*

$$d(F(x), F(y)) \leq \mu_0 d(x, y) \quad \forall x, y \in V_0, \ \forall x, y \in V_1, \tag{5.86}$$

then there exists a Cantor set $\wedge \subset V_0 \cup V_1$ such that $F : \wedge \to \wedge$ is topologically conjugate to the symbolic dynamical system $\sigma : \sum_2^+ \to \sum_2^+$, defined in the above. Consequently, F is chaotic on $\wedge$ in the sense of Devaney.

Proof. The proof is divided into three steps.

Step 1. Construct an invariant set $\wedge$ of F.

Let $K := V_0 \cup V_1$. Define the set $\wedge$ as

$$\wedge := \{x \in K : F^n(x) \in K, \ n \geq 0\}. \tag{5.87}$$

By Lemma 5.25, F has a unique fixed point in V_0 and in V_1, respectively. Thus $\wedge$ is nonempty. Obviously, $\wedge$ is an invariant set of F. We will show that $\wedge$ is a Cantor set in Step 3.

Step 2. $F : \wedge \to \wedge$ is topologically conjugate to the symbolic dynamical system $\sigma : \sum_2^+ \to \sum_2^+$.

Define a map $T : \wedge \to \sum_2^+$ as follows:

$$T(x) = s = (s_0 s_1 \cdots) \quad \text{for } x \in \wedge, \tag{5.88}$$

where $s_j = 0$ if $F^j(x) \in V_0$ and $s_j = 1$ if $F^j(x) \in V_1$. The sequence $T(x)$ is called the itinerary of x. We now show that T is homeomorphic and $T \circ F = \sigma \circ T$. Since the proof is long, it is divided into four parts.

(i) *T is bijective.* As a matter of convenience, for a subset Ω of K and $n \geq 1$, define the following set:

$$F^{-n}(\Omega) = \{x \in K \mid F^n(x) \in \Omega\}. \tag{5.89}$$

Let $s = (s_0 s_1 \cdots) \in \sum_2^+$. We can find $x \in \wedge$ such that $T(x) = s$, that is, $F^j(x) \in V_{s_j}$ for $j \geq 0$. Consider the sets

$$U_{s_0 s_1 \cdots s_n} = \{x \in K \mid F^j(x) \in V_{s_j},\ 0 \leq j \leq n\} \tag{5.90}$$

for $n \geq 0$. It is clear that $U_{s_0} = V_{s_0}$ and

$$\begin{aligned} U_{s_0 s_1 \cdots s_n} &= V_{s_0} \cap F^{-1}(V_{s_1}) \cap \cdots \cap F^{-n}(V_{s_n}) \\ &= V_{s_0} \cap F^{-1}(U_{s_1 \cdots s_n}) = U_{s_0 s_1 \cdots s_{n-1}} \cap F^{-n}(V_{s_n}) \end{aligned} \tag{5.91}$$

for $n \geq 1$. This implies that $\{U_{s_0 s_1 \cdots s_n}\}$ form a nested sequence of bounded and closed subsets of K. Now, we show by induction that they are nonempty. Obviously, $U_{s_0} = V_{s_0}$ is nonempty. From (5.91), it follows that $U_{s_0 s_1} = V_{s_0} \cap F^{-1}(V_{s_1})$. By assumption (1), we see that $F^{-1}(V_{s_1}) = V_{01} \cup V_{11}$, where V_{01} and V_{11} are nonempty closed subsets of V_0 and V_1, respectively, and $F(V_{01}) = F(V_{11}) = V_{s_1}$. Hence,

$$U_{s_0 s_1} = V_{s_0 1} \tag{5.92}$$

is nonempty. Next, suppose that $U_{s_1 \cdots s_n}$ is nonempty. It follows from (5.91) that $U_{s_1 s_2 \cdots s_n} \subset V_{s_1}$. Similarly, $F^{-1}(U_{s_1 s_2 \cdots s_n}) = V_{0n} \cup V_{1n}$, where V_{0n} and V_{1n} are nonempty close subsets of V_0 and V_1, respectively, and $F(V_{0n}) = F(V_{1n}) = U_{s_1 s_2 \cdots s_n}$. Then, from (5.91), it follows that

$$U_{s_0 s_1 \cdots s_n} = V_{s_0 n} \tag{5.93}$$

is nonempty. By induction, $U_{s_0 s_1 \cdots s_n}$ is nonempty for all $n \geq 0$.

By Lemma 5.29, in order to prove that

$$\bigcap_{n\geq 0} U_{s_0 s_1 \cdots s_n} \neq \phi, \tag{5.94}$$

it suffices to show that

$$d(U_{s_0 s_1 \cdots s_n}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.95}$$

By assumption (1), V_0 and V_1 contain infinitely many points, respectively. Let $\gamma = \max\{d(V_0), d(V_1)\}$. Then $\gamma > 0$ and

$$d(U_{s_0}) = d(V_{s_0}) \leq \gamma. \tag{5.96}$$

It follows that for all $x, y \in U_{s_0 s_1}$, $F(x), F(y) \in V_{s_1}$ and then by assumption (2),

$$d(F(x), F(y)) \geq \lambda_0 d(x, y), \tag{5.97}$$

which implies that

$$d(x, y) \leq \lambda_0^{-1} d(F(x), F(y)). \tag{5.98}$$

Hence,

$$d(U_{s_0 s_1}) \leq \lambda_0^{-1} d(V_{s_1}) \leq \lambda_0^{-1} \gamma. \tag{5.99}$$

By induction and by the definition of $U_{s_0 s_1 \cdots s_n}$,

$$d(U_{s_0 s_1 \cdots s_n}) \leq \lambda_0^{-n} \gamma, \tag{5.100}$$

which implies that (5.95) holds. Therefore, (5.94) holds and $\bigcap_{n\geq 0} U_{s_0 s_1 \cdots s_n}$ only contains one point.

Let $\bigcap_{n\geq 0} U_{s_0 s_1 \cdots s_n} = \{x\}$. Then $x \in \wedge$ and $T(x) = s$ by the definition of $U_{s_0 s_1 \cdots s_n}$. Hence, T is surjective. In addition, if $T(y) = s = (s_0 s_1 \cdots)$ for some $y \in \wedge$, then, by the definition of T, $F^j(y) \in V_{s_j}$ for $j \geq 0$, which implies that $y \in \bigcap_{n\geq 0} U_{s_0 s_1 \cdots s_n}$. Therefore, $y = x$ and consequently, the injectivity of T is proved.

(ii) *T is continuous.* Fix a point $x \in \wedge$ and let $T(x) = s = (s_0 s_1 \cdots)$. For each $\varepsilon > 0$, there exists a positive integer n such that $2^{-n} < \varepsilon$. Consider the closed set $U_{t_0 t_1 \cdots t_n}$ for all possible combinations $t_0 t_1 \cdots t_n$. It is clear that the number of these closed sets is finite and they are all disjoint by the definition of $U_{t_0 t_1 \cdots t_n}$. Now, we first show

$$d(U_{s_0 s_1 \cdots s_n}, U_{t_0 t_1 \cdots t_n}) > 0 \tag{5.101}$$

for all $s_0s_1\cdots s_n \neq t_0t_1\cdots t_n$. Let $s_0s_1\cdots s_n \neq t_0t_1\cdots t_n$. Then, there exists $0 \le j \le n$ such that $s_i = t_i$ for $0 \le i \le j-1$ and $s_j \neq t_j$. In the case of $j = 0$, $s_0 \neq t_0$. Since $U_{s_0s_1\cdots s_n} \subset V_{s_0}$ and $U_{t_0t_1\cdots t_n} \subset V_{t_0}$, it follows that

$$d(U_{s_0s_1\cdots s_n}, U_{t_0t_1\cdots t_n}) \ge d(V_{s_0}, V_{t_0}) > 0. \tag{5.102}$$

If $j \ge 1$, then $U_{s_0s_1\cdots s_n} \subset V \cap F^{-j}(V_{s_j})$ and $U_{t_0t_1\cdots t_n} \subset V \cap F^{-j}(V_{t_j})$, where $V = U_{s_0s_1\cdots s_{j-1}}$ and $s_j \neq t_j$. For any $u \in U_{s_0s_1\cdots s_n}$ and $v \in U_{t_0t_1\cdots t_n}$, $F^i(u), F^i(v) \in V_{s_i}$ for $0 \le i \le j-1$ and $F^j(u) \in V_{s_j}$, $F^j(v) \in V_{t_j}$. By assumption (3), we have

$$d(u,v) \ge \mu_0^{-1} d(F(u),F(v)) \ge \cdots \ge \mu_0^{-j} d(F^j(u),F^j(v)) \ge \mu_0^{-j} d(V_0,V_1), \tag{5.103}$$

which implies that

$$d(U_{s_0s_1\cdots s_n}, U_{t_0t_1\cdots t_n}) \ge \mu_0^{-j} d(V_0,V_1) > 0, \tag{5.104}$$

namely, (5.101) holds. Let

$$\delta = \min_{s_0s_1\cdots s_n \neq t_0t_1\cdots t_n} \{d(U_{s_0s_1\cdots s_n}, U_{t_0t_1\cdots t_n})\}. \tag{5.105}$$

Then $\delta > 0$ and for each $y \in \wedge$ with $d(x,y) \le \delta/2$, it follows that $y \in U_{s_0s_1\cdots s_n}$. Therefore, the first $n+1$ terms of $T(x)$ and $T(y)$ are the same, that is, $s_j = t_j$ for $0 \le j \le n$, where $T(y) = t = (t_0t_1\cdots t_n)$. This implies that

$$\rho(T(x),T(y)) \le \frac{1}{2^n} < \varepsilon. \tag{5.106}$$

Therefore, T is continuous.

(iii) $T^{-1} : \sum_2^+ \to \wedge$ *is continuous.* Fix a point $s = (s_0s_1\cdots) \in \sum_2^+$, and let $T^{-1}(s) = x$. Then $x \in U_{s_0s_1\cdots s_n}$ for all $n \ge 0$. For each $\varepsilon > 0$, from (5.95), it follows that there exists a positive integer N such that

$$d(U_{s_0s_1\cdots s_n}) < \varepsilon \quad \forall n \ge N. \tag{5.107}$$

Setting $\delta_0 = 1/2^N$ for all $t \in \sum_2^+$ and $\rho(t,s) < \delta_0$, we see that $t_0t_1\cdots t_N = s_0s_1\cdots s_N$ and so $y = T^{-1}(t) \in U_{s_0s_1\cdots s_N}$. This implies that

$$d(T^{-1}(t),T^{-1}(s)) = d(y,x) \le d(U_{s_0s_1\cdots s_n}) < \varepsilon. \tag{5.108}$$

Hence, T^{-1} is continuous.

(iv) $T \circ F = \sigma \circ T$. For each $x \in \wedge$, let $T(x) = s = (s_0 s_1 \cdots)$. Then $F^n(x) \in V_{s_n}$ for $n \geq 0$. By (i), $\{x\} = \bigcap_{n\geq 0} U_{s_0 s_1 \cdots s_n}$. From (5.91) and $F(V_{s_0}) \sqsupset V_{s_1}$, we get

$$\begin{aligned} F(U_{s_0 s_1 \cdots s_n}) &= F(V_{s_0} \cap F^{-1}(V_{s_1}) \cap \cdots \cap F^{-n}(V_{s_n})) \\ &= F(V_{s_0}) \cap V_{s_1} \cap \cdots \cap F^{-n+1}(V_{s_n}) \\ &= V_{s_1} \cap F^{-1}(V_{s_2}) \cap \cdots \cap F^{-n+1}(V_{s_n}) = U_{s_1 s_2 \cdots s_n}. \end{aligned} \tag{5.109}$$

Hence,

$$T(F(x)) = T\left(F\left(\bigcap_{n\geq 0} U_{s_0 s_1 \cdots s_n}\right)\right) = T\left(\bigcap_{n\geq 1} U_{s_1 s_2 \cdots s_n}\right) = (s_1 s_2 \cdots). \tag{5.110}$$

On the other hand, $\sigma(T(x)) = \sigma(s) = (s_1 s_2 \cdots)$. This implies that $(T \circ F)(x) = (\sigma \circ T)(x)$ for all $x \in \wedge$. Therefore, F and σ are topologically conjugate.

Step 3. $\wedge$ is a Cantor set.

From Step 2, $T : \wedge \to \sum_2^+$ is homeomorphic. Hence, by Lemma 5.27, $\wedge$ is compact, totally disconnected, and perfect, namely, $\wedge$ is a Cantor set.

By combining Steps 1–3 and by Lemma 5.28, the proof of Theorem 5.30 is completed. □

Next, we consider chaos generated from a continuous map in two compact subsets of a metric space. Recall from the fundamental theory of topology that a compact subset of a metric space is closed, bounded, and complete as a subspace; a closed subset of a compact space is compact; and the distance between two disjoint compact subsets of a metric space is positive. Therefore, if V_0 and V_1 are compact subsets of a metric space (X, d), (5.94) and (5.101) in Step 2 of the proof of Theorem 5.30 can be easily concluded by the compactness of $U_{s_0 s_1 \cdots s_n}$, and therefore assumption (3) in Theorem 5.30 can be dropped.

Remark 5.31. By a known result, if all assumptions of Theorem 5.30 are satisfied, then F is chaotic in the sense of Li-Yorke also.

The following is the corresponding result for chaos from a continuous map in two compact subsets of a metric space. Since the proof is trivial, based on the above illustration, it is omitted.

Theorem 5.32. *Let (x, d) be a metric space and let V_0, V_1 be two disjoint compact subsets of X. If the continuous map $F : V_0 \cup V_1 \to X$ satisfies that*

(1) $F(V_j) \supset V_0 \cup V_1$ *for* $j = 0, 1$;

(2) *there exists a constant* $\lambda_0 > 1$ *such that*

$$d(F(x), F(y)) \geq \lambda_0 d(x, y) \quad \forall x, y \in V_0, \ \forall x, y \in V_1, \tag{5.111}$$

then there exists a Cantor set $\wedge \in V_0 \cup V_1$ *such that* $F : \wedge \to \wedge$ *is topologically conjugate to the symbolic dynamical system* $\sigma : \sum_2^+ \to \sum_2^+$. *Consequently, F is chaotic on* $\wedge$ *in the sense of Devaney.*

Next, we establish two criteria of chaos by means of snap-back repellers.

Theorem 5.33. *Let* (x, d) *be a complete metric space and let* $F : X \to X$ *be a map. Assume that*

(1) *F has a regular nondegenerate snap-back repeller* $z \in X$, *that is, there exist positive constants* r_1 *and* $\lambda_1 > 1$ *such that* $F(B_{r_1}(z))$ *is open and*

$$d(F(x), F(y)) \geq \lambda_1 d(x, y) \quad \forall x, y \in \overline{B}_{r_1}(z), \tag{5.112}$$

and there exist a point $x_0 \in B_{r_1}(z)$, $x_0 \neq z$, *a positive integer m, and positive constants* δ_1 *and* γ *such that* $F^m(x_0) = z$, $B_{\delta_1}(x_0) \subset B_{r_1}(z)$, *z is an interior point of* $F^m(B_{\delta_1}(x_0))$, *and*

$$d(F^m(x), F^m(y)) \geq \gamma d(x, y) \quad \forall x, y \in \overline{B}_{\delta_1}(x_0); \tag{5.113}$$

(2) *there exists a positive constant* μ_1 *such that*

$$d(F(x), F(y)) \leq \mu_1 d(x, y) \quad \forall x, y \in \overline{B}_{r_1}(z); \tag{5.114}$$

(3) *there exists a positive constant* μ_2 *such that*

$$d(F^m(x), F^m(y)) \leq \mu_2 d(x, y) \quad \forall x, y \in \overline{B}_{\delta_1}(x_0). \tag{5.115}$$

Further, assume that F is continuous on $\overline{B}_{r_1}(z)$ *and* F^m *is continuous on* $\overline{B}_{\delta_1}(x_0)$. *Then, for each neighborhood U of z, there exist a positive integer* $n > m$ *and a Cantor set* $\wedge \subset U$ *such that* $F^n : \wedge \to \wedge$ *is topologically conjugate to the symbolic dynamical system* $\sigma : \sum_2^+ \to \sum_2^+$. *Consequently,* F^n *is chaotic on* $\wedge$ *in the sense of Devaney.*

Proof. We prove this theorem by Theorem 5.30. According to Theorem 5.30, it suffices to show that for each neighborhood U of z, there exist a positive integer $n > m$, two constants $\lambda_0 > 1$ and $\mu_0 > 0$, and two bounded and closed subsets V_0, V_1 of U with $V_0 \cap V_1 = \phi$ such that F^n is continuous on $V_0 \cup V_1$ and

$$F^n(V_j) \supset V_0 \cup V_1, \quad j = 0, 1;\ d(V_0, V_1) > 0, \tag{5.116}$$

$$d(F^n(x), F^n(y)) \geq \lambda_0 d(x, y) \quad \forall x, y \in V_0,\ \forall x, y \in V_1, \tag{5.117}$$

$$d(F^n(x), F^n(y)) \leq \mu_0 d(x, y) \quad \forall x, y \in V_0,\ \forall x, y \in V_1. \tag{5.118}$$

From assumption (1) and by an argument similar to the proof of Lemma 5.19, one can easily conclude that $F(B_r(z)) \supset B_r(z)$ and $F(\overline{B}_r(z)) \supset \overline{B}_r(z)$ for each positive constant $r \leq r_1$ and $F(D)$ is open for each open subset $D \subset B_{r_1}(z)$. We remark that this conclusion is repeatedly used in this proof.

Without loss of generality, we can suppose that $\overline{B}_{r_1}(z) \subset U$. Otherwise, we can choose an integer $\widehat{m}$, a point $\widehat{x}_0 \in B_{r_1}(z) \cap U$, and positive constants $\widehat{r} \leq r_1, \widehat{\delta}_1, \widehat{\gamma}$, and $\widehat{\mu}_2$ such that assumptions (1)–(3) hold with m, x_0, r_1, δ_1, γ, μ_2 replaced by $\widehat{m}$, $\widehat{x}_0$, $\widehat{r}_1$, $\widehat{\delta}_1$, $\widehat{\gamma}$, $\widehat{\mu}_2$, respectively. In fact, $F^{-n}(x_0) \in B_{r_1}(z) \subset W^u_{\text{loc}}(z)$ is uniquely defined for each $n \geq 1$ and $F^{-n}(x_0) \to z$ as $n \to \infty$ by Lemma 5.22. Then there exist a positive integer n_0 and a positive constant $\widehat{r}_1 \leq r_1$ such that $\widehat{x}_0 := F^{-n_0}(x_0) \in B_{\widehat{r}_1}(z) \subset U \cap B_{r_1}(z)$. It follows that $F^{n_0}(\widehat{x}_0) = x_0$, $F^{\widehat{m}}(\widehat{x}_0) = z$ with $\widehat{m} = m + n_0$, and there exists a sufficiently small positive constant $\widehat{\delta}_1$ such that $B_{\widehat{\delta}_1}(\widehat{x}_0) \subset B_{\widehat{r}_1}(z)$ and $F^i(B_{\widehat{\delta}_1}(\widehat{x}_0)) \subset B_{r_1}(z)$ for $1 \leq i \leq n_0 - 1$ and $F^{n_0}(B_{\widehat{\delta}_1}(\widehat{x}_0)) \subset B_{\delta_1}(x_0)$. Obviously, z is an interior point of $F^{\widehat{m}}(B_{\widehat{\delta}_1}(x_0))$ by (2) of Remark 5.21 and by referring to the fact that $F^i(B_{\widehat{\delta}_1}(\widehat{x}_0))$ is open for $1 \leq i \leq n_0$. From (5.112)–(5.115), it follows that for all $x, y \in \overline{B}_{\widehat{\delta}_1}(\widehat{x}_0)$,

$$d(F^{\widehat{m}}(x), F^{\widehat{m}}(y)) \geq \widehat{\gamma} d(x,y), \quad \widehat{\gamma} := \gamma\lambda_1^{n_0},$$
$$d(F^{\widehat{m}}(x), F^{\widehat{m}}(y)) \leq \widehat{\mu}_2 d(x,y), \quad \widehat{\mu}_2 := \mu_2\mu_1^{n_0}. \tag{5.119}$$

Obviously (5.112) and (5.114) hold in $\overline{B}_{\widehat{r}_1}(z)$ since $\widehat{r}_1 \leq r_1$.

The following proof is divided into three steps.

Step 1. Construct the closed set V_1 as a closed neighborhood of x_0.

Since $\lambda_1 > 1$, there exists a large integer $j \geq 1$ such that

$$\lambda_1^j \gamma > 1, \qquad \lambda_1^{-(m+j)} r_1 < \frac{d(z,x_0)}{2}. \tag{5.120}$$

From $F^m(x_0) = z$ and assumption (1), it follows that there is a small positive constant $\delta_2 \leq \delta_1$ such that

$$r_0 = d(z, \overline{B}_{\delta_2}(x_0)) > \frac{d(z,x_0)}{2}, \tag{5.121}$$

$$F^{m+i}(\overline{B}_{\delta_2}(x_0)) \subset B_{r_1}(z), \quad 0 \leq i \leq j, \tag{5.122}$$

$$F^{m+i}(\overline{B}_{\delta_2}(x_0)) \cap \overline{B}_{\delta_2}(x_0) = \phi, \quad 0 \leq i \leq j, \tag{5.123}$$

and z is an interior point of $F^{m+j}(B_{\delta_2}(x_0))$. From (5.112), (5.113), and (5.122), it follows that for all $x, y \in \overline{B}_{\delta_2}(x_0)$,

$$d(F^{m+j}(x), F^{m+j}(y)) \geq \lambda_1^j d(F^m(x), F^m(y)) \geq \lambda_1^j \gamma d(x,y), \tag{5.124}$$

which implies that F^{m+j} is expanding on $\overline{B}_{\delta_2}(x_0)$. Then $F^{m+j}(B_{\delta_2'}(x_0))$ is open for some positive constant $\delta_2' \leq \delta_2$ by Lemma 5.19. So we can suppose that $F^{m+j}(B_{\delta_2}(x_0))$ is open. It follows that $\partial F^{m+j}(B_{\delta_2}(x_0)) \subset F^{m+j}(\partial B_{\delta_2}(x_0))$.

Let $d_0 = d(z, F^{m+j}(\partial B_{\delta_2}(x_0)))$ and $l_0 = [\ln(r_1 d_0^{-1})/\ln\lambda_1] + 1$, where $d_0 < r_1$ by (5.122) and $[a]$ is the integer part of a. From (5.124), it follows that $\lambda_1^j \gamma\delta_2 \leq d_0 < r_1$.

In addition, from (5.122), it follows that

$$\overline{B}_{d_0}(z) \subset F^{m+j}(\overline{B}_{\delta_2}(x_0)) \subset B_{r_1}(z). \tag{5.125}$$

Setting $V_1' = F^{-(m+j)}(\overline{B}_{d_0}(z)) \cap \overline{B}_{\delta_2}(x_0)$, we see that V_1' is a closed subset of $\overline{B}_{\delta_2}(x_0)$, x_0 is an interior point of V_1', and

$$\begin{gathered} F^{m+i}(V_1') \subset B_{r_1}(z) \quad \text{for } 0 \leq i \leq j-1, \\ F^{m+j}(V_1') = \overline{B}_{d_0}(z) \subset B_{r_1}(z), \\ F^{m+i}(V_1') \cap V_1' = \phi, \quad 0 \leq i \leq j, \end{gathered} \tag{5.126}$$

from (5.122) and (5.123). Let $d_1 = d(z, \partial F^{m+j+1}(V_1'))$. Since $\overline{B}_{d_0}(z) \subset F(\overline{B}_{d_0}(z)) = F^{m+j+1}(V_1')$, it follows from (5.112) that

$$d_1 \geq \lambda_1 d_0. \tag{5.127}$$

To choose a suitable set V_1, the following discussion is divided into two cases of $d_1 \geq r_1$ and $d_1 < r_1$.

Case 1. $d_1 \geq r_1$. We have

$$F^{m+j+1}(V_1') \supset \overline{B}_{d_1}(z) \supset \overline{B}_{r_1}(z) \supset \overline{B}_{\delta_2}(x_0) \supset V_1'. \tag{5.128}$$

So, we set $V_1 = V_1'$ in this case.

Case 2. $d_1 < r_1$. We can continue to apply the above procedure, that is, set $V_2' = F^{-(m+j+1)}(\overline{B}_{d_1}(z)) \cap V_1'$. It is clear that V_2' is closed, $V_2' \subset V_1' \subset \overline{B}_{\delta_2}(x_0)$, x_0 is an interior point of V_2', and

$$\begin{gathered} F^{m+i}(V_2') \subset B_{r_1}(z) \quad \text{for } 0 \leq i \leq j, \\ F^{m+j+1}(V_2') = \overline{B}_{d_1}(z) \subset B_{r_1}(z), \\ F^{m+i}(V_2') \cap V_2' = \phi, \quad 0 \leq i \leq j. \end{gathered} \tag{5.129}$$

Let $d_2 = d(z, \partial F^{m+j+2}(V_2'))$. Since $\overline{B}_{d_1}(z) \subset F(\overline{B}_{d_1}(z)) = F^{m+j+2}(V_2')$, we get

$$d_2 \geq \lambda_1 d_1 \geq \lambda_1^2 d_0. \tag{5.130}$$

If $d_2 \geq r_1$, then

$$F^{m+j+2}(V_2') \supset \overline{B}_{d_2}(z) \supset \overline{B}_{r_1}(z) \supset \overline{B}_{\delta_2}(x_0) \supset V_1' \supset V_2'. \tag{5.131}$$

Hence, we set $V_1 = V_2'$ in this case. If $d_2 < r_1$, the above procedure will be continued. From (5.127) and (5.130), we see that the procedure may be continued for at most l_0 times. Suppose that the procedure is continued exactly l times, that is,

$$d_l = d(z, \partial F^{m+j+l}(V_l')) \geq r_1, \quad d_{l-1} < r_1, \tag{5.132}$$

$V_l' \subset V_{l-1}' \subset \cdots \subset V_1' \subset \overline{B}_{\delta_2}(x_0)$, x_0 is an interior point of V_l', and

$$\begin{aligned} &F^{m+i}(V_l') \subset B_{r_1}(z) \quad \text{for } 0 \leq i \leq j+l-2, \\ &F^{m+j+l-1}(V_l') = \overline{B}_{d_{l-1}}(z) \subset B_{r_1}(z), \\ &F^{m+i}(V_l') \cap V_l' = \phi, \quad 0 \leq i \leq j. \end{aligned} \tag{5.133}$$

Then, we get

$$F^{m+j+l}(V_l') \supset \overline{B}_{d_l}(z) \supset \overline{B}_{r_1}(z) \supset \overline{B}_{\delta_2}(x_0) \supset V_l'. \tag{5.134}$$

By setting $k = j + l$ and $V_1 = V_l'$, we can see that $V_1 \subset \overline{B}_{\delta_2}(x_0)$ is a closed neighborhood of x_0, and F^{m+k} satisfies the following on V_1:

$$\begin{aligned} &F^{m+i}(V_1) \subset B_{r_1}(z), \quad 0 \leq i \leq k-1, \\ &F^{m+i}(V_1) \cap V_1 = \phi, \quad 0 \leq i \leq j, \\ &F^{m+k}(V_1) \supset \overline{B}_{r_1}(z) \supset V_1. \end{aligned} \tag{5.135}$$

Furthermore, from (5.112), (5.113), and (5.135), it follows that for all $x, y \in V_1$,

$$\begin{aligned} d(F^{m+k}(x), F^{m+k}(y)) &\geq \lambda_1 d(F^{m+k-1}(x), F^{m+k-1}(y)) \\ &\geq \cdots \geq \lambda_1^k d(F^m(x), F^m(y)) \geq \lambda_1^k \gamma d(x, y) \geq \lambda_1^j \gamma d(x, y), \end{aligned} \tag{5.136}$$

where $\lambda_1^j \gamma > 1$.

Step 2. Construct V_0 as a closed neighborhood of z.

As a matter of convenience, define the following set for a subset A of $\overline{B}_{r_1}(z)$:

$$F^{-1}(A) = \{x \in \overline{B}_{r_1}(z) : F(x) \in A\}. \tag{5.137}$$

Let

$$W_0 = F^{-1}(\overline{B}_{r_1}(z)), \quad W_i = F^{-1}(W_{i-1}), \quad 1 \leq i \leq m+k-1. \tag{5.138}$$

Then

$$W_i \subset \overline{B}_{r_1}(z), \quad 0 \le i \le m+k-1,$$
$$F(W_0) \subset \overline{B}_{r_1}(z), \tag{5.139}$$
$$F(W_i) \subset W_{i-1}, \quad 1 \le i \le m+k-1.$$

It follows that

$$F^{i+1}(W_i) \subset F^i(W_{i-1}) \subset \cdots \subset F(W_0) \subset \overline{B}_{r_1}(z). \tag{5.140}$$

We claim that

$$F^{i+1}(W_i) = \overline{B}_{r_1}(z), \quad 0 \le i \le m+k-1. \tag{5.141}$$

From (5.140), it suffices to show that

$$F^{i+1}(W_i) \supset \overline{B}_{r_1}(z), \quad 0 \le i \le m+k-1. \tag{5.142}$$

For $i = 0$, we have $\overline{B}_{r_1}(z) \subset F(\overline{B}_{r_1}(z))$. Then, for each $x \in \overline{B}_{r_1}(z)$, there exists $y \in \overline{B}_{r_1}(z)$ such that $x = F(y)$. It is evident that $y \in W_0$ and $x = F(y) \in F(W_0)$, which implies that $\overline{B}_{r_1}(z) \subset F(W_0)$, so that (5.142) holds for $i = 0$. With a similar argument, one can easily show that (5.142) holds for $1 \le i \le m+k-1$.

On the other hand, from (5.112) and (5.139), it follows that for each $x \in W_i$,

$$d(F(x), z) \ge \lambda_1 d(x, z) \tag{5.143}$$

so that

$$d(x,z) \le \lambda_1^{-1} d(F(x), z) \le \lambda_1^{-1} d_s(z, W_{i-1}), \tag{5.144}$$

which implies that

$$d_s(z, W_i) \le \lambda_1^{-1} d_s(z, W_{i-1}) \le \cdots \le \lambda_1^{-(i+1)} r_1, \quad 0 \le i \le m+k-1. \tag{5.145}$$

Especially, from (5.139), (5.141), and (5.145), it follows that

$$F^i(W_{m+k-1}) \subset \overline{B}_{r_1}(z), \quad 0 \le i \le m+k-1, \tag{5.146}$$

$$F^{m+k}(W_{m+k-1}) = \overline{B}_{r_1}(z), \qquad d_s(z, W_{m+k-1}) \le \lambda_1^{-(m+k)} r_1. \tag{5.147}$$

It is clear that z is an interior point of W_i, $0 \le i \le m+k-1$, and so W_{m+k-1} is a closed neighborhood of z.

Setting $V_0 = W_{m+k-1}$, we see that V_0 is a closed neighborhood of z.

Step 3. Prove that V_0 and V_1 satisfy all the conditions (5.116)–(5.118).

It is clear that V_0 and V_1 are closed subsets of $\overline{B}_{r_1}(z)$ and, consequently, they are bounded and closed subsets of U. Set $n = m + k$. Then, we have

$$F^n(V_1) \supset V_0 \cup V_1, \qquad F^n(V_0) \supset V_0 \cup V_1 \tag{5.148}$$

by using the third relation in (5.135) and the first relation in (5.147). Therefore, the first relation in (5.116) follows. We now turn to show that

$$V_0 \cap V_1 = \phi, \qquad d(V_0, V_1) > 0. \tag{5.149}$$

From (5.147), we see that

$$d_s(z, V_0) = d_s(z, W_{m+k-1}) \leq \lambda_1^{-(m+k)} r_1 \leq \lambda_1^{-(m+j)} r_1, \tag{5.150}$$

which, together with (5.120) and (5.121), implies that $V_0 \cap V_1 \subset V_0 \cap \overline{B}_{\delta_2}(x_0) = \phi$ and

$$d(V_0, V_1) \geq d(V_0, \overline{B}_{\delta_2}(x_0)) \geq d(z, \overline{B}_{\delta_2}(x_0)) - d_s(z, V_0) \geq r_0 - \lambda_1^{-(m+j)} r_1 > 0. \tag{5.151}$$

Therefore, (5.149) holds.

Next, consider (5.117). From (5.146), we have

$$F^i(V_0) \subset \overline{B}_{r_1}(z), \quad 0 \leq i \leq m + k - 1. \tag{5.152}$$

Then, for any $x, y \in V_0$, by (5.112), we have

$$d(F^n(x), F^n(y)) \geq \lambda_1^{m+k} d(x, y) \geq \lambda_1 d(x, y). \tag{5.153}$$

Set $\lambda_0 = \min\{\lambda_1, \lambda_1^j \gamma\}$. Then, $\lambda_0 > 1$ and (5.117) follows from (5.136) and (5.153).

Finally, consider (5.118). From (5.114) and (5.152) for any $x, y \in V_0$, we have

$$d(F^n(x), F^n(y)) \leq \mu_1^n d(x, y). \tag{5.154}$$

On the other hand, from (5.114) and (5.115), and the first relation in (5.135), it follows that

$$d(F^n(x), F^n(y)) \leq \mu_1^k d(F^m(x), F^m(y)) \leq \mu_1^k \mu_2 d(x, y). \tag{5.155}$$

By setting $\mu_0 = \max\{\mu_1^n, \mu_1^k \mu_2\}$, (5.118) follows from (5.154) and (5.155). Therefore, (5.116)–(5.118) hold. By the constructions of V_0 and V_1, we see that f^n is continuous on $V_0 \cup V_1$. Hence, the proof is completed. □

By Theorem 5.32 and with an argument similar to the proof of Theorem 5.33, the following result for metric spaces with a certain compactness can be established.

Theorem 5.34. *Let (x, d) be a metric space in which each bounded and closed subset is compact. Assume that $F : X \to X$ has a regular nondegenerate snap-back repeller z, associated with x_0, m, and r as specified in Definition 5.20, F is continuous on $\overline{B}_r(z)$, and F^m is continuous in a neighborhood of x_0. Then, for each neighborhood U of z, there exist a positive integer n and a Cantor set $\wedge \subset U$ such that $F^n : \wedge \to \wedge$ is topologically conjugate to the symbolic dynamical system $\sigma : \sum_2^+ \to \sum_2^+$. Consequently, F^n is chaotic on $\wedge$ in the sense of Devaney.*

In recent years, there is growing interest in research on chaotification of dynamical systems. Now, we investigate the chaotification of partial difference equation (5.32).

Consider the controlled system

$$x_{m+1,n} = f(x_{m,n}, x_{m,n+1}) + \text{saw}_\epsilon(\mu x_{m,n}), \quad m, n \in N_0, \tag{5.156}$$

where saw_r is the classical sawtooth function, that is

$$\text{saw}_r(x) = (-1)^m(x - 2mr), \quad (2m-1)r \le x < (2m+1)r, \quad m \in Z. \tag{5.157}$$

$\mu > 0$ is controlled parameter. We want to show the condition on controlled parameter μ such that the controlled system (5.156) is chaotic in the sense of both Devaney and Li-Yorke. From Section 5.3, (5.156) is equivalent to

$$x_{m+1} = F(x_m) + \text{saw}_\epsilon(\mu x_m) \tag{5.158}$$

in the complete metric space (l^∞, d).

Equation (5.156) is chaotic in the sense of Devaney (or Li-Yorke) on $V \subset l^\infty$ if its induced system (5.158) is chaotic in the sense of Devaney (or Li-Yorke) on $V \subset l^\infty$.

Suppose that $f \in C^1$ in $[-r, r]^2$ for some $r > 0$. Denote

$$\left\{ \left| \frac{\partial f}{\partial x} \right| + \left| \frac{\partial f}{\partial y} \right| : x, y \in [-r, r] \right\}. \tag{5.159}$$

Then, $F \in C^1$ in I^∞, $I = [-r, r]$. Using Theorem 5.30, we can prove the following corollary.

Corollary 5.35. *Assume that $f \in C^1$ in $[-r, r]$ for some $r > 0$ and $f(0,0) = 0$. Then for each constant μ satisfying*

$$\mu > \mu_0 = \max\left\{ \frac{5}{2} r^{-1}\epsilon, 5(L+1) \right\}, \tag{5.160}$$

there exists a Cantor set $\Lambda \subset \overline{B}_{(5/2)\mu^{-1}\epsilon}(0) \subset l^\infty$ *such that* (5.156) *is chaotic on* Λ *in the sense of both Devaney and Li-Yorke, where* ϵ *is any given positive number.*

Consider a special case of (5.156), *given below:*

$$x_{m+1,n} = cx_{m,n}(1 - x_{m,n}) + dx_{m,n+1} + saw_\epsilon(\mu x_{m,n}), \quad m, n \geq 0. \tag{5.161}$$

It is easy to see that $f(x, y) = cx(1 - x) + dy$ *is continuously differentiable in* R^2, $f(0,0) = 0$, $f_x(x, y) = c(1 - 2x)$, *and* $f_y(x, y) = d$. *Hence*

$$|f_x(x, y)| + |f_y(x, y)| \leq 3|c| + |d|, \quad x, y \in [-1, 1]. \tag{5.162}$$

By Corollary 5.35, for each

$$\mu > \mu_0 = \max\left\{\frac{5}{2}\epsilon, 5(3|c| + |d| + 1)\right\}, \tag{5.163}$$

there exists a Cantor set $\Lambda \subset \overline{B}_{(5/2)\mu^{-1}\epsilon}(0) \subset l^\infty$ *such that the controlled system* (5.161) *is chaotic on* Λ *in the sense of both Devaney and Li-Yorke.*

5.5. Notes

First mathematical definition of chaos is from Li and Yorke [93]. Martelli et al. [112] include several definitions of chaos and their comparison; also refer to [12]. The related contents of chaos can refer Devaney [52] and Elaydi [57]. The main part of Section 5.2 is from Chen and Liu [26]. Remark 5.5 is taken from Shi et al. [131]. The contents of Section 5.3 is based on Chen et al. [27]. Banks et al. [18] point out that condition (3) in the definition of chaos in the sense of Devaney is redundant. Remark 5.11 can be seen from Huang and Ye [71]. The contents of Section 5.4 is taken from Shi and Chen [129]. Corollary 5.35 can be seen from Shi et al. [131].

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Index